When is a variety a quotient of a smooth variety by a finite group?

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Background

Any variety of the form \(X = U/G\), with \(U\) a smooth variety and \(G\) a finite group, must have quotient singularities. William Fulton posed the question, “Is every variety with quotient singularities a global quotient of a smooth variety by a finite group?”

Removing the finiteness hypothesis, the answer is known to be “yes.”

Theorem (Eddidin-Hassett-Kresch-Vistoli)

If \(X\) is a separated variety with quotient singularities over a field of characteristic 0, then \(X = U/G\), where \(U\) is a smooth variety and \(G\) is an algebraic group.

Removing the “group hypothesis,” the answer is also known to be “yes.”

Theorem (Kresch-Vistoli + above result)

If \(X\) is an irreducible quasi-projective variety with quotient singularities over a field of characteristic 0, then there is a finite flat surjection from a smooth variety \(U \to X\).

Main Result

Note that even for toric varieties, the answer to Fulton’s question is not clear since the smooth variety may not be toric. For example, there does not exist a smooth toric variety \(V\) with action of \(\mathbb{G}_m\) so that \(V/\mathbb{G}_m\) is a quotient of a smooth variety by \(\mathbb{G}_m\) acting faithfully on the fibers of \(\mathbb{G}_m\).

The ingredients of the proof are the following well-known facts.

(1) A morphism of algebraic stacks \(f: X \to Y\) is representable if and only if it induces inclusions of stabilizers at geometric points.

(2) A quotient stack \(X \cong [V/G_m]\) with \(V\) an algebraic space, is equivalent to the data of a representable morphism \(X \to BG\).

(3) A morphism to \(BGL_r\) (resp. \(BG_m\), resp. \(B\mu_n\)) is equivalent to the data of a rank \(r\) vector bundle (resp. line bundle, resp. \(n\)-torsion line bundle), and the action of the stabilizer at a geometric point is given by the induced morphism to \(GL_r\) (resp. \(G_m\), resp. \(\mu_n\)).

Key Observation

Combining (1), (2), and (3), we see that a smooth stack \(X\) is a quotient of a smooth algebraic space \(U\) by \(GL_r\) (resp. \(G_m\), resp. \(\mu_n\)) if and only if it has a vector bundle \(E\) (resp. a sum of line bundles \(E = \bigoplus L_i\), resp. a sum of torsion line bundles \(E = \bigoplus L_i\)) so that the stabilizers at geometric points act faithfully on fibers.

Sketch Proof of Theorem

Suppose \(X = V/G_m\). Let \(Y = [V/G_m]\). Then we have line bundles \(L_1, \ldots, L_r\), so that the stabilizers act faithfully on the fibers of \(\mathbb{G}_m\). Cleverly choose an integer \(r\) and sections \(s_{ij}\) of \(L_{ij}\). Let \(Y\) be the \(n\)-th root stack of \(X\) along the sections \(s_{ij}\). This stack, by its universal property, comes equipped with line bundles \(M_{ij}\) so that \(M_{ij} \cong L_{ij}\) for each \(i\). Moreover, the coarse space of \(Y\) is the same as the coarse space of \(X\), namely \(X\). Now we have that \(M_{ij} \cong L_i\) are torsion line bundles on \(Y\). Because of your clever choice of \(n\) and \(s_{ij}\), the stabilizers of \(Y\) act faithfully on the fibers of \(\bigoplus M_{ij} \cong \bigoplus L_i\). It follows that \(Y \cong [U/G]\), where \(U\) is a smooth variety and \(G = \prod \mu_n\). Thus, \(X = U/G\).

Example: \(Bl(\mathbb{P}(1, 1, 2))\)

The divisor \(D = D_1 + D_2 + D_3\) generates the class group of each torus-invariant open affine. We have that \(2 : D\) is a very ample Cartier divisor, whose polytope is shown on the right. The lattice points in the polytope are torus semi-invariant global sections of \(\mathcal{O}(2D)\). The sections \(s_a, s_b,\) and \(s_c\) corresponding to the labeled lattice points pull back to the monomials \(x_1^a x_2^b, x_1^a x_2^c,\) and \(x_1^b x_2^c\) on \(\mathbb{A}^4\), respectively. We check that the vanishing locus of \(s = s_a + s_b + s_c\) is smooth and misses the singular locus.

The variety \(W\) is given by the fan obtained by adding a ray in a new dimension and “bending down” rays 1, 2, and 3 of the original fan. The morphism \(\psi: W \to \mathbb{P}^N\) corresponds to the polytope which is a pyramid of height \(2\) on the polytope of \(D\).

The above polytope corresponds to an embedding of \(W\) into \(\mathbb{P}^N\), with the lattice points corresponding to the 19 homogeneous coordinates. Each homogeneous coordinate is acted on by \(\mu_2\), with weight given by the height of the corresponding lattice point. Let \(x_0, x_a, x_b,\) and \(x_c\) be the homogeneous coordinates corresponding to the labeled lattice points. Following the explanation described above, we let \(Y\) be the intersection of \(W\) with the hyperplane defined by \(x_0 + x_a - x_b - x_c = 0\).

This \(Y\) is a smooth variety with an action of \(\mu_2\) so that \(X \cong Y/\mu_2\).