On the ∞-Volume Limit of the Focusing Cubic Schrödinger Equation

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Abstract

We revisit the question of an invariant measure for the focusing cubic Schrödinger equation on the line. For the periodic problem the appropriate ensemble was introduced by Lebowitz, Rose, and Speer [3] and proved to be invariant under the flow by McKean [5]. These parties and others have also discussed the thermodynamic limit, though without consensus. Simulations carried out in [3] indicated the possibility of a phase transition. Similar experiments in [1] appeared to contradict that interpretation. Later, a proof was put forward in [6] that the full thermodynamic limit did not exist, suggesting a possible explanation for the disparate conclusions drawn from the numerics. Unfortunately, the latter contains an error. The main result here is that, in the infinite volume, the ensemble collapses onto the unit mass on the trivial path. Along the way sharp estimates for the partition function are established.

1 Introduction

We consider the equilibrium statistical mechanics of the cubic Schrödinger equation[1]

$$Q^\bullet = -P'' \pm (P^2 + Q^2)P = \frac{\partial H}{\partial P}$$

and

$$P^\bullet = +Q'' \mp (P^2 + Q^2)Q = -\frac{\partial H}{\partial Q}$$

taken at first on the circle $0 \le x \le L$ for which the Hamiltonian is

$$H = \frac{1}{2}\int_0^L \left[(P'(x))^2 + (Q'(x))^2\right]dx \pm \frac{1}{4}\int_0^L \left[P^2(x) + Q^2(x)\right]^2 dx\,.$$

[1] $\bullet$ signifies $\partial/\partial t$.

Communications on Pure and Applied Mathematics, Vol. LV, 1231–1248 (2002)

The problem owes its inception to Lebowitz, Rose, and Speer (LRS) [3], who introduced the petit canonical ensemble

$$(1.1)\quad e^{-H} d^{\infty}P\, d^{\infty}Q = e^{\mp(1/4)\int_0^L (P^2(x)+Q^2(x))^2}\, dx \\ \times \frac{e^{-(1/2)\int_0^L (P'(x))^2\,dx}}{(2\pi 0+)^{\infty/2}}\, d^{\infty}P \times \frac{e^{-(1/2)\int_0^L (Q'(x))^2\,dx}}{(2\pi 0+)^{\infty/2}}\, d^{\infty}Q\,.$$

The meaning of this formal object is easy to explain. The third factor signifies that Q is a "circular Brownian motion"; i.e., it is a standard Brownian motion starting at $Q(0) = m$ and conditioned to come back to m at $x = L$; this common value is distributed over the line according to the infinite measure $(2\pi L)^{-1/2}\,dm$. The second factor indicates that P is an independent copy of the same. The first factor is just a density; it ruins the independence but has a proper sense because the Brownian path is continuous. If the upper (defocusing) sign is taken, this density has the effect of controlling the far field and the ensemble may be normalized. Contrariwise, the lower (focusing) sign produces infinite total mass prompting LRS to take a microcanonical viewpoint in which the measure is restricted by fixing the constant of motion $\int_0^L (P^2 + Q^2) = N$.[2] That is, in the focusing case, one wants to consider the ensemble with partition function

$$(1.2)\qquad \mathfrak{Z}_L = \int_{\mathbb{R}^2} \mathbb{E}_c\left[e^{\frac{1}{4}\int_0^L Z^4(x)dx},\ \int_0^L Z^2(x)dx = N,\ Z(L) = c\right]dc\,.$$

Here $\mathbb{E}_\bullet$ is the mean of the planar Brownian motion $Z = (Q, P)$ starting at $\bullet \in \mathbb{R}^2$, and Z^2 is short for $Q^2 + P^2$.[3] The fact that $\mathfrak{Z}_L$ is now finite was first proven by LRS, who also showed the same holds if $\int_0^L Z^4\,dx$ is replaced by a small multiple of $\int_0^L Z^6 dx$. It is germane to note that degree 6 is critical for the nonlinear Schrödinger equation.

McKean [5] has proven the existence of the flow in and the invariance of the appropriate Gibbsian ensemble, namely, the free petit canonical measure in the defocusing case and the microcanonical measure in the focusing case. This should be compared to the work of McKean and Vaninsky [8], who discussed invariant ensembles for wave equations $Q_{tt} - Q_{xx} = -f(Q)$ of classical type, both on the circle $0 \le x < L$ and also for $L \uparrow \infty$.[4] The question raised here, then, is to identify the ∞-volume Gibbs states for cubic Schrödinger. That is, what comes out of the prescription (1.1) as the circle gets large? In the microcanonical ensemble, the proper view is to let $N = DL$ in (1.2) for some fixed density D as $L \uparrow \infty$.

[2]LRS actually took $\int_0^L Q^2 + P^2 \le N$.

[3]The peculiar but useful notation $Z(L) = c$ and the like indicate densities: $E[F(Z) = a, G(Z) = b] = (\partial^2/\partial N \partial M) E[F(Z) \le N, G(Z) \le M]|_{N=a,M=b}$.

[4]For this problem one takes $P = Q^\bullet$ and $H = \int [P^2 + (Q')^2] + \int F(Q)$ for $F(Q) = \int^Q f$. With reasonable assumptions on f, the ensemble $e^{-H}\, d^{\infty}P d^{\infty}Q$ is preserved by the flow.

In the (free) defocusing case, this is relatively simple: The petit ensemble tends to the stationary diffusion in the plane with generator

$$\mathfrak{G} = \frac{1}{2}\left(\frac{\partial^2}{\partial Q^2} + \frac{\partial^2}{\partial P^2}\right) + \left(\frac{\nabla\psi}{\psi}\right) \cdot \left(\frac{\partial}{\partial P}, \frac{\partial}{\partial Q}\right)$$

where ψ is the ground state of $\mathfrak{G}_0 = -(1/2)\triangle + (1/4)(Q^2 + P^2)^2$. One may even consider the microcanonical case here, taking first $\int_0^L Q^2 + P^2 = DL$ and then $L \uparrow \infty$. The only change in the thermodynamic limit is that a constant multiple of $Q^2 + P^2$ must be added to $\mathfrak{G}_0$, the constant being such that the limiting mean of $Q^2 + P^2$ equals D. This is in accordance with Gibbs' postulate of equivalence of ensembles (compare [7]).

The focusing case, on the other hand, is difficult; one need only note the number of varying claims in the literature. LRS carried out numerical simulations of the ensemble including temperature ($e^{-H/T}$ replaces e^{-H}). The outcome suggested a possible phase transition: the ensemble preferring radiation/solitons at low/high values of D or T. Numerical work of [1] ran contrary to this. McKean then put forward a proof that the full thermodynamic limit does not exist ([6]); that is, depending on how the circle is taken to the whole line, one sees an infinite number of limiting processes. However, [6] contains an error. The result of the present work is that the thermodynamic limit is not only unique but is in fact trivial: It is the unit mass on the zero path.

The chief instrument in the proof is an analysis of the free energy of the focusing ensemble. Taking the slightly modified partition function $\mathfrak{Z}_L$ in which $N \in L[D - \delta, D]$ for arbitrary but fixed $\delta > 0$, we prove the following asymptotic result:

$$I_D = \lim_{L\uparrow\infty} \frac{1}{L^3} \log \mathfrak{Z}_L =$$
$$\lim_{L\uparrow\infty} \frac{1}{L^3} \log \int_{\mathbb{R}^2} \mathbb{E}_c\left[e^{\frac{1}{4}\int_0^L Z^4(x)dx}, \int_0^L Z^2(x)dx \in L[D-\delta, D], Z(L) = c\right]dc$$

with I_D positive and continuous in D. While not detailed here, this appraisal is independent of temperature. That is, we encounter nothing to suggest a change of phase. More important, we find that the leading contribution to this large-deviation estimate centers on paths that make a single excursion of order L in a distance of order $1/L$. In other words, the ensemble lives near a *single* soliton regardless of the value of D. It is in fact this rather vast local concentration—the leading excursion or soliton becoming increasingly more focused with the size of the circle—which results in the collapse of the ensemble at $L = \infty$.

The paper is organized as follows: For completeness, we start in Section 2 by providing an argument that ∞-volume Gibbs states do indeed exist. In Section 3 the free energy of the ensemble is computed. The proof of the collapse occupies Section 4.

2 Existence of Gibbs States

Denote by $\mathbb{M}_L$ the (microcanonical) measure on paths defined by the partition function (1.2) with $N \in L[D-\delta, D]$. The problem posed here is to understand the weak limits (Gibbs states), if any, of this family of measures as $L \uparrow \infty$. Since the rotation invariance of the circular Brownian motion is inherited by $\mathbb{M}_L$, it is plain that if anything is to come out in the infinite volume, it must be stationary. Thus, in order to understand $\mathbb{M}_\infty$, it is enough to examine a fixed segment of the path while L gets large. In brief, for F a "short" test function depending on $Z(x')$ only for $0 \le x' \le x < L$, say, we need to consider the limit(s) of

$$\mathbb{M}_L[F] = \mathfrak{Z}_L^{-1} \int\limits_{\mathbb{R}^2} \mathbb{E}_c\Big[e^{\frac{1}{4}\int_0^L Z^4(x')dx'} F[Z(x') : 0 \le x' \le x], \\ \int_0^L Z^2(x')dx' \in L[D-\delta, D], Z(L) = c \Big] dc$$

as $L \uparrow \infty$. We begin by noting that the microcanonical fiat $\int_0^L Z^2(x)dx \le LD$ provides enough control for the question to make sense.

First, consider the density

$$p(x, a, b, I) = \mathbb{E}_a\left[e^{\frac{1}{4}\int_0^x Z^4(x')dx'}, Z(x) = b, \int_0^x Z^2(x')dx' = I \right].$$

It is the fundamental solution of the hypoelliptic equation

$$\frac{\partial p}{\partial x} = \frac{1}{2}\triangle p + \frac{1}{4}c^4 p - c^2\frac{\partial p}{\partial I}$$

relative to either $c = a$ or $c = b$, and, as such, is smooth and positive as soon as x and I are greater than zero.[5] Next, by Chapman-Kolmogorov the mean value (2.1) is rewritten as follows:

$$\mathbb{M}_L[F] = \mathfrak{Z}_L^{-1} \int_{D-\delta}^{D} \int_0^{LN} \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} p(L-x, b, a, LN-I) \\ \times \mathbb{E}_a\left[e^{\frac{1}{4}\int_0^x Z^4(x')dx'} F(Z), \int_0^x Z^2(x')dx' = I, Z(x) = b \right] da\, db\, dI\, dN.$$

The point is that all that changes in the above for $L \uparrow \infty$ is the ratio $\mathfrak{Z}_L^{-1} p(L - x, b, a, LN - I)$, and thus the problem goes over into understanding the large L behavior of this object.

As for that, it is a point of comfort that the family of measures

$$m_L(da, db, dI) = \mathfrak{Z}_L^{-1} \int_{D-\delta}^{D} p(x, a, b, I)p(L-x, b, a, LN-I)da\, db\, dI\, dN$$

[5] Krylov [2] covers this.

on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+$ is tight: Just observe

$$\mathfrak{Z}_L^{-1} \int_{D-\delta}^{D} \int_0^{LN} \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left(\frac{a^2+b^2}{2}\right)$$

$$p(x,a,b,I)p(L-x,b,a,LN-I)da\,db\,dI\,dN$$

$$= \mathfrak{Z}_L^{-1} \int_{D-\delta}^{D} \int\limits_{\mathbb{R}^2} c^2 p(L,c,c,LN)dc\,dN$$

$$= \mathbb{M}_L[Z^2(0)] = \mathbb{M}_L\left[\frac{1}{L}\int_0^L Z^2(x)dx\right] \le D$$

and

$$\mathfrak{Z}_L^{-1} \int_{D-\delta}^{D} \int_0^{LD} \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} Ip(x,a,b,I)p(L-x,b,a,LN-I)da\,db\,dI\,dN =$$

$$\mathbb{M}_L\left[\int_0^x Z^2(x')dx'\right] \le xD\,.$$

Next, p being strictly positive, $\mathfrak{Z}_L^{-1}\int_{D-\delta}^{D} p(L-x,b,a,LN-I)da\,db\,dI\,dN$ is tight as well. Limit points $h(da, db, dI)$ of the latter even possess smooth densities. The equation for p implies that h is a weak solution of $0 = \partial h/\partial x + \frac{1}{2}\triangle h + \frac{1}{4}c^4 p + c^2\partial h/\partial I$ again for $c = a$ or $c = b$; hypoellipticity then dictates that $h = h(x, a, b, I)$, a smooth function away from $x = 0$ and $I = 0$.

The outcome of the preceding remarks is that, by taking $L \uparrow \infty$ over some *suitable* subsequence, there is a Gibbs state $\mathbb{M}_\infty$ described by

$$\mathbb{M}_\infty[F] = \int_0^\infty \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} E_a\left[e^{\frac{1}{4}\int_0^x Z^4(x')dx'}F(Z),\, \int_0^x Z^2(x')dx' = I,\, Z(x) = b\right]$$

$$\times h(x,a,b,I)da\,db\,dI\,;$$

the equality guaranteed by choosing F bounded and vanishing for large a, b, and I. In this way one may pick off limiting joint densities of $Z_\infty(x)$, $Z_\infty(x')$, $\int_x^{x'} Z^2(c)dc$, and the like from which to reconstruct the limit process. Our claim that $\mathbb{M}_\infty$ is trivial then amounts to the family of limiting h's reducing to some sort of delta measures at the origin in a, b, I. The error in [6] was in fact a mistaken proof that any h had to be strictly positive as soon as x and I were greater than zero.

3 Computation of the Free Energy

In examining the thermodynamic limit of $\mathbb{M}_L$, getting a handle on the partition function is a necessary first step. We have the following:

THEOREM 3.1 *The microcanonical partition function*

$$\mathfrak{Z}_L = \int_{\mathbb{R}^2} \mathbb{E}_c\Big[e^{\frac{1}{4}\int_0^L Z^4(x)dx}, \int_0^L Z^2(x)dx \in L[D-\delta, D], Z(L) = c\Big]dc$$

satisfies

$$\lim_{L\uparrow\infty} \frac{1}{L^3} \log \mathfrak{Z}_L = \sup_{\int_{-\infty}^{\infty} |f(x)|^2\,dx = D} \left\{ \frac{1}{4}\int_{-\infty}^{\infty} |f(x)|^4\,dx - \frac{1}{2}\int_{-\infty}^{\infty} |f'(x)|^2\,dx \right\} \equiv I_D\,.$$

The quantity I_D makes sense because of the well-known interpolation inequality $\|f\|_4^4 \le c\|f'\|_2\|f\|_2^3$. It is continuous and positive as a function of D; no phase transition is exhibited at this level.

3.1 Proof for the Lower Bound

It is more convenient to extend the integration from $-L/2$ to $L/2$. Fix $f \in C_\downarrow^\infty(\mathbb{R})$ with $D - \delta < \int_{-\infty}^{\infty} |f|^2\,dx < D$ and define $f_L(x) = Lf(Lx)$, noting that $\int_{-\infty}^{\infty} |f_L|^2\,dx = L\int_{-\infty}^{\infty} |f|^2\,dx$. Now, for small enough $\varepsilon > 0$,

$$\begin{aligned}
\mathfrak{Z}_L &= \int_{\mathbb{R}^2} \mathbb{E}_c\Big[e^{\frac{1}{4}\int_{-L/2}^{L/2} Z^4(x)dx}, \int_{-L/2}^{L/2} Z^2(x)dx \in L[D-\delta, D], Z(L) = c\Big]dc \\
&\ge \int_{\mathbb{R}^2} \mathbb{E}_c\Big[e^{\frac{1}{4}\int_{-L/2}^{L/2} Z^4(x)dx}, \|Z - f_L\|_\infty < \varepsilon, Z(L) = c\Big]dc \\
&\ge \exp\left[\frac{1}{4}\int_{-L/2}^{L/2} |f_L|^4\,dx - \sup_{\|\phi - f_L\|_\infty \le \varepsilon} \left| \int_{-L/2}^{L/2} |\phi|^4\,dx - \int_{-L/2}^{L/2} |f_L|^4\,dx \right|\right] \\
&\qquad \times \int_{|c|\le\varepsilon} \mathbb{P}_c\big[\|Z - f_L\|_\infty \le \varepsilon, Z(L) = c\big]dc\,, \qquad (3.1)
\end{aligned}$$

where the last inequality may be simplified by employing the estimate

$$\left| \int_0^L \phi^4 - \int_0^L \psi^4 \right| \le \eta\left[\int_0^L \psi^4 + L\right]$$

with η going to zero with $\|\phi - \psi\|_\infty$.

Next, we may assume that restricting f_L to $[-L/2, L/2]$ satisfies $f_L(-L/2) = f_L(L/2)$ by rotation invariance. After this, an application of the Cameron-Martin formula yields

$$\begin{aligned}
\mathfrak{Z}_L &\ge \exp\left[\frac{1}{4}(1-\eta)\int_{-L/2}^{L/2} |f_L|^4\,dx - \frac{1}{2}\int_{-L/2}^{L/2} |f_L'|^2\,dx - \eta L\right] \\
&\qquad \times \int_{\|c\|\le\varepsilon} \mathbb{E}_c\Big[e^{\int_{-L/2}^{L/2} f_L'(x)\cdot dZ(x)}, \|Z\|_\infty \le \varepsilon, Z(L) = c\Big]dc\,. \qquad (3.2)
\end{aligned}$$

The exponential in the last expectation may then be treated by an integration by parts. This, along with an unscaling of the f_L, produces

$$
\begin{aligned}
\frac{1}{L^3}\log \mathfrak{Z}_L \geq \frac{1}{4}(1-\eta)\int_{-L^2/2}^{L^2/2} |f|^4\,dx - \frac{1}{2}\int_{-L^2/2}^{L^2/2} |f'|^2\,dx - \frac{1}{L^2}\eta \\
-\frac{1}{L}\int_{-L^2/2}^{L^2/2} |f''|dx \\
+\frac{1}{L^3}\log\left(\int_{\|c\|\leq\varepsilon} \mathbb{P}_c[\|Z\|_\infty \leq \varepsilon, Z(L)=c]dc\right).
\end{aligned}
\tag{3.3}
$$

Note that our assumptions on f imply control of $\int_{-\infty}^{\infty}|f''|dx$ and $\int_{-\infty}^{\infty}|f'|^2\,dx$. As for the last term above, the well-known density of absorbing Brownian motion shows that

$$
\int_{\|c\|\leq\varepsilon} \mathbb{P}_c[\|Z\|_\infty \leq \varepsilon, Z(L)=c]dc \simeq \varepsilon^{-1}\exp\left[-2\pi^2 L/\varepsilon^2\right],
\tag{3.4}
$$

allowing the conclusion

$$
\liminf_{L\uparrow\infty}\frac{1}{L^3}\log \mathfrak{Z}_L \geq (1-\eta)\frac{1}{4}\int_{-\infty}^{\infty}|f|^4\,dx - \frac{1}{2}\int_{-\infty}^{\infty}|f'|^2\,dx\,.
$$

Of course, ε was arbitrary and taking it $\downarrow 0$ makes $\eta \downarrow 0$ as well. The result then follows by optimizing the choice of f and noticing, via a scaling, that the supremum over the set $\int_{-\infty}^{\infty}|f|^2\,dx \leq D$ is the same as that over $\int_{-\infty}^{\infty}|f|^2\,dx = D$.

3.2 Proof for the Upper Bound

For clarity, the proof is carried out for

$$
\mathbb{E}_{00}\left[\exp\left[\left(\frac{1}{4}\right)\int_0^L Z^4(x)dx\right], \int_0^L Z^2(x)dx \leq LD\right]
$$

with tied (00) boundary conditions; how to handle the periodicity is explained at the end. The proof is divided into seven steps.

Step 1

The first step proves the needed estimate that for any positive λ,

$$
\limsup_{L\uparrow\infty}\frac{1}{L^3}\log\mathbb{E}_{00}\left[e^{\lambda\int_0^L Z^4(x)dx}, \int_0^L Z^2(x)dx \leq LD\right] \leq C < \infty
\tag{3.5}
$$

with constant C depending on λ and D. With $\mathrm{BM}_\bullet$ denoting the one-dimensional Brownian mean and $|Z|_\infty$ and the like denoting running maxima, the expectation in (3.5) is bounded above by

$$\mathbb{E}_{00}\left[e^{\lambda LD|Z|_\infty^2}, \int_0^L Z^2(x)dx \le LD\right] \le \\ e^{\lambda L^3 D} + \left\{\mathrm{BM}_{00}\left[e^{\lambda LD|Q|_\infty^2}, \int_0^L Q^2(x)dx \le LD, |Q|_\infty \ge \frac{L}{2}\right]\right\}^2.$$

This turns the problem into that of estimating

$$p(L,z) \equiv \mathrm{BM}_{00}\left[\max_{0\le x\le L}|Q|(x) \ge Lz, \int_0^L Q^2 \le LD\right] \\ \le \mathrm{BM}_{00}\left[\exists x, k : \left|x - \frac{k}{m}\right| \le \frac{1}{m} \text{ and } \left|Q(x) - Q\left(\frac{k}{m}\right)\right| \ge \frac{Lz}{4}\right]$$

with integer k and $m = \lceil \frac{Lz^2}{4D} \rceil$. For the inequality, note that $\int_0^L Q^2(x)dx \le LD$ implies that $\mathrm{meas}\{x \le L : |Q(x)| \ge Lz/2\} \le 4D/Lz^2$. Thus, in order for $|Q|_\infty$ to exceed Lz, Q must make an excursion of size $Lz/2$ in an interval of at most $4D/Lz^2$ in length.

Now L, and so also m, is large, and we may assume both k/m and x are at a fixed distance, say 1, from either 0 or L. But, if $\mathcal{A}$ is an event measurable over the field $\{Q(x), 0 \le x \le L-1\}$, then with $p(x,a,b) =$ the Brownian transition density, one has

$$\mathrm{BM}_{00}[\mathcal{A}] = \mathrm{BM}_0\left[\frac{p(1, Q(L-1), 0)}{p(L,0,0)}, \mathcal{A}\right] \\ = \sqrt{L}\,\mathrm{BM}_0\left[e^{-Q^2(L-1)/2}, \mathcal{A}\right] \le \sqrt{L}\,\mathrm{BM}_0[\mathcal{A}].$$

This, coupled with the (bridge) equivalence in law $Q(x) \sim Q(L-x)$, provides the bound

$$p(L,z) \le Lm\mathrm{BM}_0\left[\max_{0\le x\le 1/m}|Q|(x) \ge \frac{Lz}{4}\right] \\ = \sqrt{\frac{L^3m^3}{2\pi}} \int_{Lz/4}^\infty e^{-mc^2/2}\,dc \le \frac{1}{\sqrt{D}} L^2 z^2 e^{-L^3z^4/(128D)},$$

the equality being an instance of the reflection principle. From this point, the proof of (3.5) is completed via a straightforward Laplace integral.

Step 2

This step introduces what is meant by a soliton in the present computation. Along with the typical passage time $\mathfrak{m}_a$ of Z to a level $|Z| = a$, we bring in the following loop times. For $0 < \delta \ll 1$, let

$$\begin{aligned} \mathfrak{m}_1 &= \inf\left\{x > 0 : |Z|(x) = 2\delta L\right\}, \\ \mathfrak{m}_2 &= \inf\left\{x > \mathfrak{m}_1 : |Z|(x) = \delta L\right\}, \\ &\ \vdots \\ \mathfrak{m}_{2k} &= \inf\left\{x > \mathfrak{m}_{2k-1} : |Z|(x) = \delta L\right\}, \\ &\ \vdots \end{aligned} \tag{3.6}$$

These are the "down-crossings" from the circle of radius $2\delta L$ to the smaller circle of radius δL and should be viewed as solitons. Notice that in each down-crossing the path performs an excursion with length of order L. We will show that the contribution to the desired expectation comes entirely from the down-crossing set $DC = \bigcup[\mathfrak{m}_{2k-1}, \mathfrak{m}_{2k}]$.

Step 3

The third step proves that with overwhelming probability there are only a finite number of down-crossings, and each has width of order $1/L$.[6] More precisely, given the events

$$\mathcal{A}_K = \left\{Z : \forall k, \mathfrak{m}_{2k} - \mathfrak{m}_{2k-1} \in \left[\frac{1}{KL}, \frac{K}{L}\right]\right\}$$

and

$$\mathcal{B}_N = \left\{Z : \max\{k : \mathfrak{m}_{2k} \le L\} \le N\right\},$$

we prove their complements are superexponentially rare with rate L^3.

As to $\mathcal{A}_K^c$, the condition $\int_0^L Z^2(x)dx \le LD$ implies $\sum(\mathfrak{m}_{2k} - \mathfrak{m}_{2k-1}) \le D/\delta^2 L$, and so the length of any down-crossing is at most order $1/L$. The length is not much smaller either: For large positive K and k such that $\mathfrak{m}_{2k} < L$, we can estimate

$$\begin{aligned} \mathbb{P}_{00}\left[\mathfrak{m}_{2k} - \mathfrak{m}_{2k-1} \le \frac{1}{KL}\right] &\le L\mathbb{P}_0\left[\mathfrak{m}_{2k} - \mathfrak{m}_{2k-1} \le \frac{1}{KL}\right] \\ &= L\mathbb{P}_{2\delta L}\left[\inf\left\{x > 0 : |Z|(x) \le \delta L\right\} \le \frac{1}{KL}\right] \\ &\le CL^{5/2}e^{-K\delta^2L^3/2}. \end{aligned} \tag{3.7}$$

The first inequality makes use of the (previously used) estimate on the Radon-Nykodym derivative of the Brownian bridge with respect to the free Brownian

[6] Later it will be proven that one sees only a single down-crossing for $L \uparrow \infty$.

motion and the fact that $\mathfrak{m}_{2k} - \mathfrak{m}_{2k-1} \le 1/2K$ implies that either $\mathfrak{m}_{2k-1}$ or $L - \mathfrak{m}_{2k}$ exceeds $1/KL$. The rest is plain. In a similar manner, we see that the probability of the path satisfying $\int_0^L Z^2(x)dx \le LD$ and experiencing N down-crossings before $x = L$ is no larger than

$$\begin{aligned}
&\mathbb{P}_{00}\left[\sum_1^N (\mathfrak{m}_{2k} - \mathfrak{m}_{2k-1}) \le \frac{D}{\delta^2 L}, \mathfrak{m}_{2N} \le L\right] \\
&\quad \le LN\mathbb{P}_0\left[\sum_1^N (\mathfrak{m}_{2k} - \mathfrak{m}_{2k-1}) \le \frac{D}{\delta^2 L}\right] + \mathbb{P}_{00}\left[\mathfrak{m}_1 \le \frac{1}{NL}, \mathfrak{m}_{2N} \ge L - \frac{1}{NL}\right] \\
&\quad \le LN \max_{\theta:\sum\theta_k \le 1} \prod_1^N \mathbb{P}_{2\delta L}\left[\mathfrak{m}_{\delta L} \le \frac{D\theta_k}{\delta^2 L}\right] + L^{3/2}\exp[-L^3 N\delta^2] \\
&\quad \le c_1 L^2 \exp\left[-L^3 N^2 \frac{\delta^4}{2D}\right] + c_2 L^{3/2}\exp[-L^3 N\delta^2].
\end{aligned}$$

In line 3 we made use of the fact that the down-crossings are independent under the free Brownian mean and our estimate from (3.7). This proves the claim for $\mathcal{B}_N^c$.

Step 4

The fourth step amplifies the accomplishments of steps 1 and 3. Knowing that the logarithm of the expectation grows at most like L^3 (step 1), an application of the Cauchy-Schwartz inequality along with step 3 shows that, with K and N taken sufficiently large,

$$\begin{aligned}
&\mathbb{E}_{00}\left[e^{\frac{1}{4}\int_0^L Z^4(x)dx}, \int_0^L Z^2(x)dx \le LD\right] \\
&\quad \simeq \mathbb{E}_{00}\left[e^{\frac{1}{4}\int_0^L Z^4(x)dx}, \int_0^L Z^2(x)dx \le LD, \mathcal{A}_K, \mathcal{B}_N\right] \\
&\quad \le 2KL^2 e^{L^3 D\delta^2}\mathbb{E}_0\left[e^{\frac{1}{4}\int_{DC} Z^4(x)dx}, \int_{DC} Z^2(x)dx \le DL, \mathcal{A}_K, \mathcal{B}_N\right].
\end{aligned}$$

The second line has two ingredients. First, observe that, off of DC, $|Z| \le 2\delta L$ and so $\int_0^L Z^4 \le 4\delta^2 L^2 \int_0^L Z^2 \le 4D\delta^2 L^3$. Also, the Bridge mean has been replaced by the free Brownian mean by the usual trick (on $\mathcal{A}_K$ the entirety of DC lies a distance $1/2KL$ from either endpoint).

Next, one only makes things larger by replacing $\int_{DC} Z^2(x)dx \le LD$ with the event $\bigcap\{\int_{\mathfrak{m}_{2k-1}}^{\mathfrak{m}_{2k}} Z^2(x)dx \le \theta_k LD\}$ and then maximizing over θ satisfying $\sum\theta_k \le$

1. The Markov property and the isotropic nature of the Brownian path can then be used to split the mean over the individual down-crossings. That is,

$$(3.8)\quad \mathbb{E}_{00}\Big[e^{\frac{1}{4}\int_0^L Z^4(x)dx}, \int_0^L Z^2(x)dx \le LD\Big] \le \gamma(L)e^{L^3D\delta^2} \times \max_{n\le N} \sup_{\theta:\sum_1^n \theta_k \le 1} \prod_{k=1}^n \Xi_k(L, D, K)$$

with $\gamma(L)$ a polynomial factor in L and, for any k,

$$\Xi_k(L, D, K) = \mathbb{E}_0\Big[\exp\Big\{\frac{1}{4}\int_{\mathfrak{m}_1}^{\mathfrak{m}_2} Z^4(x)dx\Big\}, \int_{\mathfrak{m}_1}^{\mathfrak{m}_2} Z^2(x)dx \le L\theta_k D, \frac{1}{KL} \le \mathfrak{m}_2 - \mathfrak{m}_1 \le \frac{K}{L}\Big].$$

Step 5

The fifth step turns attention to $\Xi_k(L, D, K)$, subjecting it to a series of scalings in order to bring out the correct rate L^3. First we perform a change of variables in both integrals inside the expectation, sending $\int_{\mathfrak{m}_1}^{\mathfrak{m}_2} Z^4(x)dx$ into

$$\frac{1}{L}\int_{L\mathfrak{m}_1}^{L\mathfrak{m}_2} Z^4(x'/L)dx'$$

and $\int_{\mathfrak{m}_1}^{\mathfrak{m}_2} Z^2(x)dx$ into

$$\frac{1}{L}\int_{L\mathfrak{m}_1}^{L\mathfrak{m}_2} Z^2(x'/L)dx'.$$

This move is followed by invoking the equivalence in law $\sqrt{L}Z(x/L) \sim Z(x)$ to produce

$$\begin{aligned}(3.9)\quad \Xi_k(L, D, K) &= \mathbb{E}_0\Big[\exp\Big\{\frac{1}{4L}\int_{L\mathfrak{m}_1}^{L\mathfrak{m}_2} Z^4(\frac{x}{L})dx\Big\},\\ &\qquad \int_{L\mathfrak{m}_1}^{L\mathfrak{m}_2} Z^2(\frac{x}{L})dx \le L\theta_k D, \frac{1}{KL} \le \mathfrak{m}_2 - \mathfrak{m}_1 \le \frac{K}{L}\Big]\\ &= \mathbb{E}_0\Big[\exp\Big\{\frac{L^3}{4}\int_{\hat{\mathfrak{m}}_1}^{\hat{\mathfrak{m}}_2} \Big[\frac{1}{L^{3/2}}Z(x)\Big]^4 dx\Big\},\\ &\qquad \int_{\hat{\mathfrak{m}}_1}^{\hat{\mathfrak{m}}_2} \big[\frac{1}{L^{3/2}}Z(x)\big]^2 dx \le \theta_k D, \frac{1}{K} \le \hat{\mathfrak{m}}_2 - \hat{\mathfrak{m}}_1 \le K\Big].\end{aligned}$$

The hats over $\hat{\mathfrak{m}}_1$ and $\hat{\mathfrak{m}}_2$ indicate new loop times from the disk of radius 2δ down to that of radius δ, which arise since $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are transformed under the scaling.

For example,

$$\begin{aligned}\mathfrak{m}_1 &= \inf\left\{x > 0 : |Z|(x) \ge 2\delta L\right\}\\ &\sim \inf\left\{x > 0 : \frac{1}{\sqrt{L}}|Z|(Lx) \ge 2\delta L\right\} = \frac{1}{L}\inf\left\{x' > 0 : \frac{1}{L^{3/2}}|Z|(x') \ge 2\delta\right\},\end{aligned}$$

which we define to be $\frac{1}{L}\hat{\mathfrak{m}}_1(L^{-3/2}Z)$. The scaling of $\mathfrak{m}_2$ is much the same.

Notice that in (3.9), each appearance of the path u is multiplied by the factor $L^{-3/2}$. Thus, we bring in $\mathbb{E}_\bullet^{L^{-3}}$ to denote the mean of the "slow" Brownian motion with L^{-3} diffusion coefficient[7] and re-express things as in

$$\begin{aligned}(3.10)\quad \Xi_k(L, D, K) &= \mathbb{E}_0^{L^{-3}}\bigg[\exp\left\{\frac{L^3}{4}\int_{\hat{\mathfrak{m}}_1}^{\hat{\mathfrak{m}}_2} Z^4(x)dx\right\},\\ &\qquad\qquad \int_{\hat{\mathfrak{m}}_1}^{\hat{\mathfrak{m}}_2} Z^2(x)dx \le \theta_k D, \frac{1}{K} \le \hat{\mathfrak{m}}_2 - \hat{\mathfrak{m}}_1 \le K\bigg]\\ &= \mathbb{E}_\bullet^{L^{-3}}\bigg[\exp\left\{\frac{L^3}{4}\int_0^{\mathfrak{m}_\delta} Z^4(x)dx\right\},\\ &\qquad\qquad \int_0^{\mathfrak{m}_\delta} Z^2(x)dx \le \theta_k D, \frac{1}{K} \le \mathfrak{m}_\delta \le K\bigg].\end{aligned}$$

In the second line, $\bullet$ indicates any fixed starting point on the outer circle of radius 2δ; the equality is another consequence of the Markov property and the isotropy of each Brownian functional involved.

Step 6

The sixth step exploits that we now have a more or less standard Laplace integral: The expectation in (3.10) has the rate L^3 in front of both the potential $\int Z^4$ and the energy $\int |Z'|^2$.

As is typical in such estimates (see, for example, [9]) we introduce Z_n, the piecewise linear interpolant of Z: The points $(0, Z(0)), (1/n, Z(1/n)), \dots,$ are joined by a broken line. Note that

$$\begin{aligned}&\mathbb{P}_\bullet^{L^{-3}}\left[\|Z(x) - Z_n(x)\|_\infty \ge \varepsilon \text{ for } x \le K\right] \le\\ &\qquad 4Kn\mathrm{BM}_0\left[\max_{0\le x\le 1/n} Q(x) \ge \frac{L^{3/2}\varepsilon}{2\sqrt{2}}\right] \le \frac{16Kn^2}{\varepsilon L^{3/2}} e^{-n\varepsilon^2 L^3/16},\end{aligned}$$

and also that $\mathbb{P}_\bullet^{L^{-3}}(\|Z_n\|_\infty \ge M) \le C\exp[-L^3M^2/K]$. Therefore, with n and M large, (3.10) may be further restricted to the set

$$\mathcal{C}_{n,\varepsilon,M} = \left\{Z : \|Z - Z_n\|_\infty \le \varepsilon,\ \|Z_n\|_\infty \le M\right\}.$$

[7]In other words, the energy $\exp\{-\frac{1}{2}\int |Z'|^2\}$ is replaced by $\exp\{-\frac{L^3}{2}\int |Z'|^2\}$.

On this set the path satisfies

$$\int_0^{\mathfrak{m}} Z_n^2(x)dx \le \int_0^{\mathfrak{m}} Z^2(x)dx + \varepsilon \int_0^{\mathfrak{m}} |Z|(x)dx + \varepsilon^2 \mathfrak{m} \le \theta_k D + c_1\varepsilon\,,$$

and similarly $\int_0^{\mathfrak{m}} Z_n^4(x)dx \le \int_0^{\mathfrak{m}} Z^4(x)dx + c_2\varepsilon$ for constants c_1 and c_2. We may then continue to overestimate as in $\Xi_k(L, D, K) \le$

$$\mathbb{E}_\bullet^{L^{-3}}\left[e^{\frac{L^3}{4}\int_0^{\mathfrak{m}_\delta} Z^4(x)dx}, \int_0^{m_\delta} Z^2(x)dx \le \theta_k D, \mathfrak{m}_\delta \le K, \mathcal{C}_{n,\varepsilon,M}\right] \le$$

$$e^{L^3 c_1\varepsilon}\mathbb{E}_\bullet^{L^{-3}}\left[e^{\frac{L^3}{4}\int_0^{\mathfrak{m}_\delta} Z_n^4(x)dx}, \int_0^{\mathfrak{m}_\delta} Z_n^2(x)dx \le \theta_k D + c_2\varepsilon, \mathfrak{m}_\delta \le K\right],$$

up to exponentially small errors. Next, an approximation to the correct rate function finally makes an appearance by inserting the energy term $\int |Z_n'|^2$ as follows:

$$\Xi_k(L, D, K) \le \exp\left[L^3 I_{\theta_k D}(\varepsilon, \alpha, K) + L^3 c_1\varepsilon\right] \times \mathbb{E}_\bullet\left[\exp\left\{\frac{\alpha}{2} n \sum_{i=0}^{nK-1}\left|Z\left(\frac{k+1}{n}\right) - Z\left(\frac{k}{n}\right)\right|^2\right\}\right] \tag{3.11}$$

where $\alpha < 1$ and

$$I_N(\varepsilon, \alpha, K) = \sup_{\int_{-K/2}^{K/2}|f|^2\,dx \le N + c_2\varepsilon}\left\{\frac{1}{4}\int_{-K/2}^{K/2}|f|^4\,dx - \frac{\alpha}{2}\int_{-K/2}^{K/2}|f'|^2\,dx\right\}.$$

Now, the expectation in (3.11) reduces to $(E[e^{\alpha X/2}])^{nK}$ for X distributed as the sum of two independent squared standard Gaussians; this is finite for all $\alpha < 1$ and certainly independent of L. It follows that $I_{\theta_k D}(\varepsilon, \alpha, K)$ dominates

$$\limsup_{L\uparrow\infty} L^{-3}\log \Xi_k(L, D, K)$$

and by letting $\varepsilon \downarrow 0$ and $\alpha \uparrow 1$ afterward, you also conclude that

$$\limsup_{L\uparrow\infty}\frac{1}{L^3}\log \Xi_k(L, D, K) \le \sup_{\int_{-\infty}^{\infty}|f|^2\,dx = \theta_k D}\left\{\frac{1}{4}\int_{\infty}^{\infty}|f|^4\,dx - \frac{1}{2}\int_{\infty}^{\infty}|f'|^2\,dx\right\},$$

which is just $I_{\theta_k D}$.

Finally, δ was also arbitrary, and so

$$\begin{aligned}
&\limsup_{L\to\infty}\frac{1}{L^3}\log \mathbb{E}_{00}\left[\exp\left[\frac{1}{4}\int_0^L Z^4(x)dx, \int_0^L Z^2(x)dx \le LD\right]\right.\\
&\quad \le \max_{n\le N}\max_{\theta:\sum_1^n\theta_k=1}\sum_{k=1}^n I_{\theta_k D}\\
&\quad = \max_{n\le N}\max_{\theta:\sum_1^n\theta_k=1}\sum_{k=1}^n \theta_k\left[\sup_{\int_{-\infty}^{\infty} f^2\,dx = D}\left\{\frac{\theta_k}{4}\int_{-\infty}^{\infty}|f|^4\,dx - \int_{-\infty}^{\infty}|f'|^2\,dx\right\}\right],
\end{aligned}$$

which is less than I_D as is seen by setting each $\theta_k = 1$ inside the brackets of the last line. The maximization actually takes place with strict inequality at $n = 1$; asymptotically there is just one soliton.

Step 7

The last step puts back the periodic boundary condition. Replacing E_{00} with E_{cc} for a fixed c clearly won't change the above; one may even allow $c = o(L)$. Next note that the measure of the set for which $|Z| \geq \sqrt{L}$ is at most order one. So, by rotation invariance,

$$\begin{aligned}\mathfrak{Z}_L &\leq L \int_{\mathbb{R}^2} E_{cc}\Big[e^{\frac{1}{4}\int_0^L Z^4(x)}\, dx, \int_0^L Z^2(x)dx \leq LD, \\ &\qquad |Z|(x) \leq \sqrt{L} \text{ for } 0 \leq x \leq 1 \Big] dc \\ &\leq L \int_{|c|\leq\sqrt{L}} E_{cc}\Big[e^{\frac{1}{4}\int_0^L Z^4(x)dx}, \int_0^L Z^2(x)dx \leq LD \Big] dc\,,\end{aligned}$$

from which it is plain that demonstrating the asymptotics for the bridge is sufficient. The proof is finished.

4 Proof of the Collapse of the Ensemble

The computation of the free energy in the previous section indicates that, for $L \uparrow \infty$, the leading-order paths under $\mathbb{M}_L$ live near a *single* increasingly focused (dimension L by $1/L$) soliton. It is now proven that this results in collapse at $L = \infty$. The idea is that while a typical path wants to peak, rotation invariance implies the position this peak is, in a sense, equally distributed over the circle and is therefore lost when viewing a short segment of the path as $L \uparrow \infty$. This is summarized in the following statement.

THEOREM 4.1 *The unique thermodynamic limit of the $\mathbb{M}_L$ ensemble is the trivial measure that places unit mass on the zero path.*

PROOF: The qualitative argument of the preceding paragraph is made rigorous by introducing the probability measure $\mathcal{M}_L$ on the space measures on the unit circle induced by $\mathbb{M}_L$ by the map from paths $Z(\cdot) \to \nu(Z, dx)$ defined by

$$\nu([a,b]) = \frac{1}{L}\int_{La}^{Lb} |Z|^2(x)dx\,.$$

Note that $\nu([0,1]) = D$ almost surely under $\mathcal{M}_L$. The main step of the proof is to demonstrate that as $L \uparrow \infty$, $\mathcal{M}_L$ concentrates its mass on $D\delta_{x^*}(x)$, with x^* uniformly distributed over the circle. The concentration of any ν onto a point mass follows from the fact proven next that, for any $\gamma > 0$ and interval $[a,b]$,

$$m_{\gamma,L} = \mathcal{M}_L\big[\nu([a,b]) \geq \gamma, \nu([a,b]^c) \geq \gamma\big] \to 0 \tag{4.1}$$

in the large L limit. The intuition is of course that anything contrary to (4.1) requires the appearance of more than one soliton. That the location of the point mass should be uniformly distributed is again due to rotation invariance.

Without loss of generality, one may take $[a, b] = [0, x]$. Next, by arguments similar to those in the upper bound of Theorem 3.1, we may reduce the problem from periodic to free boundary conditions. This yields

$$
\begin{aligned}
m_{\gamma,L} \le \mathfrak{Z}_L^{-1} \int_{\mathbb{R}^2} E_{cc}\Big[e^{\frac{1}{4}\int_0^L Z(x')dx'}, \int_0^{Lx} Z^2(x')dx' \ge L\gamma,& \\
\int_{Lx}^{L} Z^2(x')dx' \ge L\gamma, \int_0^L Z^2(x') \le LD \Big] dc& \\
\le c(L)\mathfrak{Z}_L^{-1} \max_{\gamma \le N \le D-\gamma} \Big\{ \mathbb{E}_0\Big[e^{\frac{1}{4}\int_0^{Lx} Z^4(x')dx'} \int_0^{Lx} Z^2(x')dx' \le LN \Big]& \\
\times \mathbb{E}_0\Big[e^{\frac{1}{4}\int_0^{L-Lx} Z^4(x')dx'} \int_0^{Lx} Z^2(x')dx' \le LD - LN \Big]\Big\}&
\end{aligned} \tag{4.2}
$$

with $c(L)$ a polynomial factor in L.

Again looking to the proof of the upper bound in Theorem 3.1, we see that equations (3.9) through (3.11) may be compacted to obtain the following estimate:

$$
\begin{aligned}
\mathbb{E}_0\Big[\exp\Big\{ \frac{1}{4}\int_0^{Lx} Z^4(x')dx' \Big\}, \int_0^{Lx} Z^2(x')dx' \le LN \Big] \le& \\
\varepsilon^{-nK} \exp\big[L^3 I_N(\varepsilon) + L^3 c_1(\delta^2 + \varepsilon)\big] + O\big(e^{-L^3 c_2(\varepsilon,n,\delta,K)}\big)&
\end{aligned} \tag{4.3}
$$

with

$$
I_N(\varepsilon) = \sup_{\int_{-\infty}^{\infty} |f(x)|^2\,dx = N + c_3\varepsilon} \Big\{ \frac{1}{4}\int_{-\infty}^{\infty} |f(x)|^4\,dx - \frac{1-\varepsilon}{2}\int_{-\infty}^{\infty} |f'(x)|^2\,dx \Big\}.
$$

Here the parameters satisfy $\varepsilon > 0$, $\delta > 0$, $n\varepsilon^2 \gg 1$, and $K\delta^2 \gg 1$ but are otherwise arbitrary. The constants c_1 and c_3 are fixed independently of L or the above parameters, and c_2 is made as large as one wishes by taking $n\varepsilon^2$ so. Furthermore, it is clear that

$$
I_N(\varepsilon) \le I_N(0) + c_4(\varepsilon) = I_N + c_4(\varepsilon) \tag{4.4}
$$

with $\lim_{\varepsilon\downarrow 0} c_4(\varepsilon) = 0$. Of course, the same upper bound applies to the final expectation in (4.6) with $D - N$ replacing N.

Next, the partition function must be controlled from below. Turning to the proof of Theorem 3.1 once more, this time to the lower bound, we may obtain

$$
\log \mathfrak{Z}_L \ge L^3 I_D - c_5 L^2 \tag{4.5}
$$

with c_5 a constant depending only on D. This refined lower bound is obtained by reworking the same argument. The main difference is to include the fact (see [4]) that the maximizer of $I(f) = (1/4)\int |f|^4\,dx - (1/2)\int |f'|^2\,dx$ over $\int |f|^2\,dx = D$ is actually attained for a smooth positive and rapidly decaying function that we will denote f^*. What we had before is then sharpened by integrating around a tube of width $L^{-1/2}$ about $f_L^* = L\sqrt{1-1/L}\,f^*(Lx)$. Note first that

$$\|Z - f_L^*\|_\infty \le L^{-1/2} \quad \text{implies} \quad LD - 2D \le \int_0^L Z^2(x)dx \le LD\,,$$

and so, following equations (3.1) through (3.4) with f^* as our test function, we have

$$\log \mathfrak{Z}_L \ge L^3\left[\frac{1}{4}\int_{-L^2/2}^{L^2/2} |f^*|^4\,dx - \frac{1}{2}\int_{-L^2/2}^{L^2/2} |(f^*)'|^2\,dx\right] - L^2\int_{-L^2/2}^{L^2/2} |(f^*)''|dx$$
$$- \sup_{\phi:\|\phi - f_L^*\|_\infty \le L^{-1/2}} \left[\int_{-L^2/2}^{L^2/2} \big||\phi|^4 - |f_L^*|^4\big|dx\right] - 2\pi^2 L^2 - \log L\,.$$

The bound (4.5) then follows from checking that

$$\frac{1}{4}\int_{-L^2/2}^{L^2/2} |f^*(x)|^4\,dx - \frac{1}{2}\int_{-L^2/2}^{L^2/2} |(f^*(x))'|^2\,dx \ge I_D - o(L^{-3})$$

and

$$\int_{-L^2/2}^{L^2/2} \big||\phi|^4 - |f_L^*|^4\big|dx \le c_6 L^{3/2} \quad \text{for } \phi \text{ satisfying } \|\phi - f_L^*\|_\infty \le L^{-1/2}\,.$$

Both are immediate consequences of the rapid decay of f^*.

Estimates (4.3) and (4.5) are then combined to reveal that

$$\text{(4.6)} \quad m_{\gamma,L} \le c_7(\varepsilon,\delta)\exp\Big[L^3\Big(\max_{\gamma\le N\le D-\gamma}(I_N + I_{D-N}) - I_D\Big) + L^3 c_8(\varepsilon,\delta) + c_5 L^2\Big]$$

with $c_7(\varepsilon,\delta)$ finite for ε and δ positive and $\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} c_8(\varepsilon,\delta) = 0$. Now, a simple scaling argument shows that

$$\max_{\gamma\le N\le D-\gamma}(I_N + I_{D-N}) - I_D < 0$$

for $\gamma > 0$. Therefore, given γ we may choose δ and ε small enough to force the right-hand side of (4.6) to vanish as $L \uparrow \infty$. This completes the proof of (4.1).

To finish, split [0, 1) into $1/\varepsilon$ disjoint intervals $\{I_k\}$ of length $\varepsilon > 0$. An application of Hölder's inequality and the properties of $\mathcal{M}_L$ proven above yield

$$\mathbb{M}_L[|Z|(0)] = \sum_{k=1}^{1/\varepsilon} \mathbb{M}_L\left[\frac{1}{L}\int_{LI_k} |Z|(x)dx\right]$$
$$\le \sqrt{\varepsilon}\sum_{k=1}^{1/\varepsilon} \mathbb{M}_L\left[\sqrt{\frac{1}{L}\int_{LI_k} Z^2(x)dx}\right]$$

$$= \sqrt{\varepsilon} \sum_{k=1}^{1/\varepsilon} \left(\mathcal{M}_L\left[\sqrt{\nu(I_k)}, \nu(I_k) > \varepsilon^2\right] + \mathcal{M}_L\left[\sqrt{\nu(I_k)}, \nu(I_k) < \varepsilon^2\right] \right)$$

$$\leq \sqrt{\varepsilon} \sum_{k=1}^{1/\varepsilon} \left(\sqrt{D} \mathcal{M}_L[\nu(I_k) \geq \varepsilon^2] + \varepsilon \right) \to \sqrt{\varepsilon}\left(1 + \sqrt{D}\right).$$

Since ε is arbitrary, $\limsup_{L\uparrow\infty} \mathbb{M}_L[|Z|(0)] = 0$. Appealing once more to the microcanonical fiat $\mathbb{M}_L[Z^2(0)] = D$, one concludes $\mathbb{M}_\infty[|Z|(0)] = 0$ for any limiting $\mathbb{M}_\infty$. Finally, as any thermodynamic limit point is stationary, it must be that $\mathbb{M}_\infty$ places all its mass on the trivial path. The proof is finished. □

Remarks. 1. The collapse of the microcanonical ensemble should prove a general feature for systems with an indefinite Hamiltonian, the connection being superexponential growth of the total mass. For focusing NLS we have seen that this rate is [volume]^3. In KdV the field is one-dimensional with Hamiltonian $H = \frac{1}{3}\int_0^L Q^3(x)dx + \int_0^L (Q'(x))^2\,dx$, which, when exponentiated, is again balanced by conditioning on the constant of motion $\int_0^L Q^2 = LD$. An argument completely analogous to the above shows that the logarithm of the total mass grows like $L^{4/3}$ and leads to the collapse of the ensemble.

2. The next task would be to study fluctuations about the "law of large numbers" constituted by the present collapse. That is, for some $\gamma = \gamma_L \uparrow \infty$ with $L \uparrow \infty$, one wishes to compute $\lim_{L\uparrow\infty} \mathbb{M}_L[F(\gamma Q, \gamma P)]$ for a suitable class of test functions. Such a calculation is beyond us. However, preliminary results on discrete caricatures of the present ensemble lead us to believe that at a scaling of $\gamma = \sqrt{L}$ one has fluctuations resembling a white noise.

Acknowledgments. It's a pleasure to thank my advisor, H. P. McKean, for suggesting the problem, his active participation in this work, and ongoing support. Thanks as well to Professor S. R. S. Varadhan for many useful discussions. Finally, I am grateful to the patient referee for many helpful suggestions, particularly those behind the proof of Theorem 4.1.

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Received January 2001.
Revised May 2002.