(IN CLASS) MIDTERM SOLUTIONS

1. For each x, let $a = a_{x,n} = \inf\{y \le x : |f(x) - f(y)| < 1/n\}$ and $b = b_{x,n} = \sup\{y \ge x : |f(x) - f(y)| < 1/n\}$. Set $I_x^n = (a, b)$ as long as one of a or b does not equal x, otherwise I_x^n is empty. Take $\mathcal{O}_n = \cup_x I_x^n$ (which is open), then $G = \cap_{n=1}^{\infty} \mathcal{O}_n$ is the desired G_δ set = points of continuity of f.

2. Take any partition Γ of $[a, b]$ and consider the refinement Γ' arrived at by including the points $\{a + \delta, a + 2\delta, a + 3\delta, \dots a + m\delta\}$ for m the largest integer less than $(b - a)/\delta$ and δ chosen so that

$$\sum_i |f(x_{i+1}) - f(x_i)| \le 1 \text{ for any } \{x_i\} \text{ with } \sum_i |x_{i+1} - x_i| \le \delta.$$

Absolute continuity insures we can do this.

With $\Gamma' = \{a = y_0 < y_1 < \cdots < y_n = b\}$ note:

$$V_{f,\Gamma} \le V_{f,\Gamma'} = \sum_{k=0}^{m} \sum_{y_i \in [a+k\delta,(a+(k+1)\delta)\wedge b]} |f(y_{i+1}) - f(y_1)| \le m + 1,$$

and so f is bounded variation.

To show that bounded variation does not imply absolute continuity, take the interval $[0, 1]$ and

$$f(x) = \begin{cases} 0 \text{ for } x \in [0, 1/2), \\ 1 \text{ for } x \in [1/2, 1]. \end{cases}$$

The variation of f is 1, and you cannot make $f(y) - f(x)$ any smaller than 1 if $y \ge 1/2$ and $x < 1/2$ no matter how small $y - x$.

3. We have $\{f_k\}$ with $0 \le f_k \le f_{k+1}$ a.e. for all k, and $f_k \stackrel{m}{\to} f$. Take any subsequence $\{f_k'\}$ (which also converge in measure to f) and produce a further subsequence $\{f_{k''}\}$ with $f_{k''} \to f$ a.e. But now $\{f_{k''}\}$ satisfies all conditions of the MCT and so

$$x_{k''} \equiv \int_E f_{k''} \to \int_E f \equiv x.$$

In other words, any subsequence of $\{x_k\} = \{\int_E f_k\}$ contains a further subsequence which converges to $\int_E f$ and so we have convergence of the full sequence $\int_E f_k \to \int_E f$.