

# Erratum: Diffusion at the random matrix hard edge

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As stated, Theorem 2 of [2] is valid only for  $a \geq 0$ . While this restriction does not effect the main result of [2], we still would like to communicate the correct statement of Theorem 2 and its proof.

The starting point remains Theorem 1 of the same paper. There it was proved that the scaled limiting points of the  $(\beta, a)$ -Laguerre ensemble<sup>1</sup> corresponds to the eigenvalues of  $\mathfrak{G}_{\beta, a}$  where  $-\mathfrak{G}_{\beta, a}$  generates the diffusion process with random speed and scale measures

$$m(dx) = e^{-(a+1)x - \frac{2}{\sqrt{\beta}}b(x)} dx, \quad s(dx) = e^{ax + \frac{2}{\sqrt{\beta}}b(x)} dx.$$

Here  $x \mapsto b(x)$  is a standard Brownian motion. More precisely, the reciprocal of the  $(\beta, a)$ -Laguerre points converge to the ordered eigenvalues of the almost surely trace class integral operator

$$(\mathfrak{G}_{\beta, a})^{-1}\psi(x) = \int_0^\infty \int_0^{x \wedge y} s(dz)\psi(y)m(dy), \text{ for } \psi \in L^2[\mathbb{R}_+, m].$$

The corresponding boundary conditions for an eigenfunction  $f$  may be read off from the above. That one must have  $\psi(0) = 0$  is clear, but it was a misunderstanding of the boundary “at infinity” which led to the error in question. For  $\mathfrak{G}_{\beta, a}$ ,  $+\infty$  is entrance, not exit if  $a \geq 0$ , while it is entrance *and* exit when  $a \in (-1, 0]$  (see [1] for definitions). In fact, the theorem in question remains correct for  $a \geq 0$  (though for reasons slightly different than stated). For  $a < 0$ , the process generated by  $\mathfrak{G}_{\beta, a}$  can reach  $+\infty$ , and this has consequences not addressed in [2].

Consider now the “Riccati diffusion” for  $\mathfrak{G}_{\beta, a}$ , given by the Itô equation

$$dp(x) = \frac{2}{\sqrt{\beta}}p(x)db(x) + \left( (a + \frac{2}{\beta})p(x) - p^2(x) - \lambda e^{-x} \right) dx, \quad (1)$$

( $x$  is the time parameter). The process  $p$  may be begun at  $+\infty$ , which it leaves instantaneously, and there is a positive probability of explosion to  $-\infty$ . Note also, if  $p(x) = 0$  at some  $x < \infty$ , then  $p(x') < 0$  for all  $x < x'$ .

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<sup>1</sup>This is the measure with density proportional to  $\prod_{j < k} |\lambda_j - \lambda_k|^\beta \times \prod_{k=0}^{n-1} \lambda_k^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2}\lambda_k}$  on  $(\mathbb{R}_+)^n$ . Here  $\beta > 0$  and  $a > -1$ .

With  $\Lambda_0(\beta, a) < \Lambda_1(\beta, a) < \dots$  the eigenvalues of  $\mathfrak{G}_{\beta, a}$ , the corrected Theorem 2 ([2]) reads as follows.

**Theorem.** *Let  $P_{\infty, x}$  denote the law induced by  $p(\cdot; \beta, a, \lambda)$  started at  $+\infty$  at time  $x$ , and restarted at  $+\infty$  and time  $\mathbf{m}$  upon any  $\mathbf{m} < \infty$ ,  $p(\mathbf{m}) = -\infty$ . Then,*

$$\begin{aligned} P(\Lambda_0(\beta, a) > \lambda) &= P_{\infty, 0}(p \text{ never hits } 0), \\ P(\lambda_k(\beta, a) < \lambda) &= P_{\infty, 0}(p \text{ hits } 0 \text{ at least } k + 1 \text{ times}). \end{aligned}$$

The proof still comes down to Sturm oscillation. The difference between  $a \geq 0$  and  $-1 < a < 0$  is connected to the next simple fact pointed out to us by O. Zeitouni.

**Claim.** *With  $m_c$  the passage time to position  $c$ ,  $P(\mathbf{m}_{-\infty} < \infty | \mathbf{m}_0 < \infty) = 1$  whenever  $a \geq 0$ .*

In words, when  $a \geq 0$  the process  $x \mapsto p(x)$  will hit  $-\infty$  with probability one once it hits 0. Hence we also have the following.

**Corollary.** *Let  $a \geq 0$ , and set  $\nu_x(dc) = P_{\infty, x}(\mathbf{m}_{-\infty} \in dc)$ . Then,  $P(\Lambda_0(\beta, a) > \lambda) = \nu_0(\{\infty\})$  and  $P(\lambda_k(\beta, a) < \lambda) = \int_{\mathbb{R}^{k+1}} \nu_0(dx_1) \nu_{x_1}(dx_2) \cdots \nu_{x_k}(dx_{k+1})$ .*

This corollary is just how Theorem 2 ([2]) was stated, erroneously, for all  $a > -1$ . Effectively, when  $a \geq 0$  we can think of there being a Dirichlet condition at  $+\infty$ . Said better, for  $a \geq 0$  the half-line eigenvalue problem is the  $L \uparrow \infty$  limit of the problem on  $[0, L]$  with either Dirichlet or Neumann conditions at  $L$  (and Dirichlet at 0). For  $-1 < a < 0$  however the boundary point at  $+\infty$  must be viewed as Neumann – it can only be approximated by a sequence of problems on  $[0, L]$  with a Neumann condition at  $L$ . In terms of  $x \mapsto p(x)$ , this results in counting passages to 0 (Neumann) rather than to  $-\infty$  (Dirichlet).

Finally note that Theorem 3 of [2] (which is a direct corollary of Theorem 2) is unaffected, this being a statement about  $a \uparrow +\infty$ .

*Proof of the theorem.* Bring in the approximating operator  $\mathfrak{G}_{\beta, a}^L$  ( $0 < L < \infty$ ) defined via

$$(\mathfrak{G}_{\beta, a}^L)^{-1} \psi(x) = \int_0^\infty s_L(x, y) \psi(y) m(dy), \quad s_L(x, y) = \left[ \int_0^{x \wedge y} s(dz) \right] 1_{x, y \in [0, L]}, \quad (2)$$

acting on  $\psi \in L^2([0, L], m)$ . One may readily check that any solution to  $\psi(x) = \lambda(\mathfrak{G}_{\beta, a}^L)^{-1} \psi(x)$  must satisfy  $\psi(0) = 0$  and  $\psi'(L) = 0$ . That is, this defines the eigenvalue problem for  $\mathfrak{G}_{\beta, a}$ , cut down to  $[0, L]$  with Dirichlet/Neumann conditions at 0 and  $L$ .

In the manner of Lemma 12 of [2] one may check that  $(\mathfrak{G}_{\beta, a}^L)^{-1} \rightarrow (\mathfrak{G}_{\beta, a})^{-1}$  in trace norm, which provides convergence of the ordered eigenvalues. As a bit of amplification, in Lemma 12 of [2] we had applied the argument to the approximating  $\tilde{\mathfrak{G}}_{\beta, a}^L$  prescribing Dirichlet

eigenvalues at both 0 and  $L$ . In that case the integral kernel  $s_L(x, y)$  (with respect to  $m$ ) from (2) is changed to

$$\tilde{s}_L(x, y) = \left[ \int_0^{x \wedge y} s(dz) \right] \times \left[ \frac{\int_{x \vee y}^L s(dz)}{\int_0^L s(dz)} \right] 1_{x, y \in [0, L]}.$$

The crux of the problem is that  $(\tilde{\mathfrak{G}}_{\beta, a}^L)^{-1} \rightarrow (\mathfrak{G}_{\beta, a})^{-1}$  only for  $a \geq 0$ .

Differentiating both sides of  $\psi(x) = \lambda \int_0^\infty s_L(x, y) \psi(y) m(dy)$  leads to the same system for  $x \mapsto (\psi(x), \psi'(x))$  found in [2] (see equation (3.1) in that reference). Further,  $x \mapsto p(x) = \psi'(x)/\psi(x)$ , sensible away from the zeros of  $\psi$ , again solves the stochastic differential equation (1).

With this setup,  $\lambda$  is an eigenvalue of  $\mathfrak{G}_{\beta, a}^L$  only if  $(\psi(x, \lambda), \psi'(x, \lambda))$  satisfying the differential system with initial condition  $(0, 1)$  at  $x = 0$  comes to  $(\cdot, 0)$  at  $x = L$ . In terms of  $p$ ,  $\lambda$  is an eigenvalue if  $p(L, \lambda) = 0$  (after possible passages to  $-\infty$  and subsequent “re-starts”). As  $\lambda$  increases, one sees that the zeros of  $p$  on  $[0, L]$  move from right to left, additional zeros appearing at  $L$ . At this point, the proof proceeds exactly as in [2], with the understanding that the passages of  $p$  to 0, not to  $-\infty$ , comprises the eigenvalue counting function.  $\square$

## References

- [1] ITÔ, K., MCKEAN, H.P. *Diffusion processes and their sample paths*. Springer-Verlag. Berlin-Heidelberg-New York, 1974.
- [2] RAMÍREZ, J., RIDER, B. (2009) Diffusion at the random matrix hard edge. *Communications in Math. Phys.* **288**, no. 3, 887-906.

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