

Volume Growth and General Rate Quantization on Grassmann Manifolds^{*}

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Abstract—The Grassmann manifold $\mathcal{G}_{n,p}(\mathbb{L})$ is the set of all p -dimensional planes (through the origin) in the n -dimensional Euclidean space \mathbb{L}^n , where \mathbb{L} is either \mathbb{R} or \mathbb{C} . This paper considers an unequal dimensional quantization in which a source in $\mathcal{G}_{n,q}(\mathbb{L})$ is quantized through a code in $\mathcal{G}_{n,p}(\mathbb{L})$, where p and q are not necessarily the same. The analysis for unequal dimensional quantization is based on the volume of a metric ball in $\mathcal{G}_{n,q}(\mathbb{L})$ whose center is in $\mathcal{G}_{n,p}(\mathbb{L})$. Our chief result is to show that as n, p, q and the square radius approach infinity with constant ratios, the volume of a metric ball “grows” as $\exp(-n^2V(1+o(1)))$ for a computable constant $V \geq 0$. This result is stronger than our previous volume formula which is only valid when the radius is at most one. The tools behind the present result include large deviation techniques and equilibrium measure ideas from potential theory. Based on the volume growth formula, the rate distortion tradeoff is precisely quantified in our asymptotic region. Finally, we prove that random codes are asymptotically optimal in probability.

Index Terms—the Grassmann manifold, rate distortion tradeoff, MIMO communications

I. INTRODUCTION

The Grassmann manifold $\mathcal{G}_{n,p}(\mathbb{L})$ is the set of all p -dimensional planes (through the origin) in the n -dimensional Euclidean space \mathbb{L}^n , where \mathbb{L} is either \mathbb{R} or \mathbb{C} . It forms a compact Riemann manifold of real dimension $\beta p(n-p)$, where $\beta = 1$ when $\mathbb{L} = \mathbb{R}$ and $\beta = 2$ when $\mathbb{L} = \mathbb{C}$. The Grassmann manifold is a useful analysis tool for multi-antenna communications (also known as multiple-input multiple-output (MIMO) communication systems). The capacity of non-coherent MIMO systems at high signal-to-noise ratio (SNR) region was derived by analysis in the Grassmann manifold [1]. The well known unitary space time codes for MIMO systems can be viewed as codes in the Grassmann manifold. Further, for coherent MIMO systems with finite rate feedback, the quantization of eigenchannel vectors is related to the quantization on the Grassmann manifold [2]–[5].

This paper studies the unequal dimensional quantization on the Grassmann manifold. Roughly speaking, a quantization is a representation of a source: it maps an element in $\mathcal{G}_{n,q}(\mathbb{L})$ (the source) into a subset $\mathcal{C} \subset \mathcal{G}_{n,p}(\mathbb{L})$, which is often discrete and referred to as a *code*. While most works assume that $p = q$ [6]–[10], we are interested in a more general case where p may not necessarily equal to q ; thus the term unequal dimensional quantization. The performance limit of quantization is given

by the so called rate distortion tradeoff. Let the source be randomly distributed and define a *distortion* metric between elements in $\mathcal{G}_{n,q}(\mathbb{L})$ and $\mathcal{G}_{n,p}(\mathbb{L})$. The rate distortion tradeoff is described by the minimum average distortion achievable for a given code size, or equivalently the minimum code size required to achieve a particular average distortion. This paper will quantify the rate distortion tradeoff for unequal dimensional quantization.

The quantization problem has been extensively studied recently. Most works discuss the equal dimensional quantization ($p = q$) only. The Rankin bound in $\mathcal{G}_{n,p}(\mathbb{R})$ is obtained in [6] when the code size is large. The Gilbert-Varshamov and Hamming bounds on $\mathcal{G}_{n,p}(\mathbb{L})$ are calculated in [7] when p is fixed and n is asymptotically large. The distortion rate tradeoff for $p = 1$ is quantified in [2], [3] by analysis in the Grassmann manifold and in [5] using high resolution quantization theory. Different from above approaches, our papers [9], [10] extend the traditional equal dimensional quantization to the unequal dimensional quantization. However, the results are only valid when the quantization rate is high. Specifically, the characterization of the rate distortion tradeoff is closely related to the volume calculation of a metric ball in the Grassmann manifold. Let $B(\delta)$ denote a metric ball of radius δ in the Grassmann manifold. In [9], [10], an explicit volume formula is derived only when the radius $\delta \leq 1$. As a result, the rate distortion tradeoff is explicitly characterized only when the quantization rate is high.

The main contribution of this paper is to derive a volume growth formula and therefore the rate distortion tradeoff for the general case. Specifically:

- 1) As n, p, q , and the square radius δ approach infinity with constant ratios, the volume of a metric ball is $\exp(-n^2V(1+o(1)))$ for a computable constant $V \geq 0$. Towards this explicit formula, the technique employed is quite different from that in our previous papers [9], [10]. There, the volume calculation relies on the evaluation of Selberg’s second generalization of the Beta integral. The derivation here rests on large deviation techniques and equilibrium measure ideas from potential theory.
- 2) Based on this volume growth formula, the exact asymptotic rate distortion tradeoff is quantified as n, p, q and the average distortion scale linearly. The derived tradeoff holds for arbitrary n^2 -normalized rate (see Section IV for details on the rate normalization), thus the term

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general rate. This result is stronger than that in [8]–[10] which is only valid for high rate quantization. Further, random codes are proved to be asymptotically optimal in probability.

II. PRELIMINARIES

A. Metric and Measure on the Grassmann manifold

For the sake of applications [2]–[4], the projection Frobenius metric (*chordal distance*) and the invariant measure are employed throughout this paper. Without loss of generality, we assume that $p \leq q$ (see (4) for details). For any two planes $P \in \mathcal{G}_{n,p}(\mathbb{L})$ and $Q \in \mathcal{G}_{n,q}(\mathbb{L})$, we define the principal angles and the chordal distance between P and Q as follows. Let $\mathbf{u}_1 \in P$ and $\mathbf{v}_1 \in Q$ be the unit vectors such that $|\mathbf{u}_1^\dagger \mathbf{v}_1|$ is maximal. Inductively, let $\mathbf{u}_i \in P$ and $\mathbf{v}_i \in Q$ be the unit vectors such that $\mathbf{u}_i^\dagger \mathbf{u}_j = 0$ and $\mathbf{v}_i^\dagger \mathbf{v}_j = 0$ for all $1 \leq j < i$ and $|\mathbf{u}_i^\dagger \mathbf{v}_i|$ is maximal. The principal angles are defined as $\theta_i = \arccos |\mathbf{u}_i^\dagger \mathbf{v}_i|$ for $i = 1, \dots, p$ [6], [11]. The chordal distance between P and Q is given by

$$d_c(P, Q) \triangleq \sqrt{\sum_{i=1}^p \sin^2 \theta_i}. \quad (1)$$

The invariant measure μ on $\mathcal{G}_{n,p}(\mathbb{L})$ is the Haar measure on $\mathcal{G}_{n,p}(\mathbb{L})$. Let $O(n)$ and $U(n)$ be the groups of $n \times n$ orthogonal and unitary matrices respectively. Let $\mathbf{A} \in O(n)$ when $\mathbb{L} = \mathbb{R}$, or $\mathbf{A} \in U(n)$ when $\mathbb{L} = \mathbb{C}$. For any measurable set $\mathcal{M} \subset \mathcal{G}_{n,p}(\mathbb{L})$ and arbitrary \mathbf{A} , μ satisfies

$$\mu(\mathbf{A}\mathcal{M}) = \mu(\mathcal{M}).$$

The invariant measure defines the uniform/isotropic distribution on $\mathcal{G}_{n,p}(\mathbb{L})$ [11].

B. Quantization and the Rate Distortion Tradeoff

This paper addresses an unequal dimensional quantization problem. Let \mathcal{C} be a finite size discrete subset of $\mathcal{G}_{n,p}(\mathbb{L})$ (also known as a code). An unequal dimensional quantization is a mapping from the $\mathcal{G}_{n,q}(\mathbb{L})$ to the set \mathcal{C} , $\mathfrak{q} : \mathcal{G}_{n,p}(\mathbb{L}) \rightarrow \mathcal{C}$, where p and q are not necessarily the same integer. Without loss of generality, we assume $p \leq q$. We are interested in quantifying the *rate distortion tradeoff*. Assume that a source $Q \in \mathcal{G}_{n,q}(\mathbb{L})$ is isotropically distributed. Define the distortion measure as the square of the chordal distance $d_c^2(\cdot, \cdot)$. Then the distortion associated with a quantization \mathfrak{q} is defined as $D \triangleq \mathbb{E} [d_c^2(Q, \mathfrak{q}(Q))]$. For a given code $\mathcal{C} \subset \mathcal{G}_{n,p}(\mathbb{L})$, the optimal quantization to minimize the distortion is given by $\mathfrak{q}(Q) = \arg \min_{P \in \mathcal{C}} d_c(P, Q)$. The corresponding distortion

is $D(\mathcal{C}) = \mathbb{E} \left[\min_{P \in \mathcal{C}} d_c^2(P, Q) \right]$. The rate distortion tradeoff can be described by the *distortion rate function*: the infimum achievable distortion given a code size K

$$D^*(K) = \inf_{\mathcal{C}: |\mathcal{C}|=K} D(\mathcal{C}), \quad (2)$$

or the *rate distortion function*: the minimum required code size to achieve a given distortion D

$$K^*(D) = \inf_{D(\mathcal{C}) \leq D} |\mathcal{C}|. \quad (3)$$

III. VOLUME CALCULATION ON GRASSMANN MANIFOLDS

This section derives the closed-form formula for volume growth of a metric ball in the Grassmann manifold. This formula is the essential tool to quantify the rate distortion tradeoff in Section IV.

The volume of a metric ball can be expressed as a multivariate integral. Let μ and d_c be the invariant measure and the chordal distance on the Grassmann manifold. For any given $P \in \mathcal{G}_{n,p}(\mathbb{L})$ and $Q \in \mathcal{G}_{n,q}(\mathbb{L})$ ($p \leq q$ assumed w.l.o.g.), define metric balls (with square radius δ) by

$$B_P(\delta) = \left\{ \hat{Q} \in \mathcal{G}_{n,q}(\mathbb{L}) : d_c^2(P, \hat{Q}) \leq \delta \right\}$$

and

$$B_Q(\delta) = \left\{ \hat{P} \in \mathcal{G}_{n,p}(\mathbb{L}) : d_c^2(\hat{P}, Q) \leq \delta \right\}.$$

It can be shown that

$$\mu(B_P(\delta)) = \mu(B_Q(\delta)) \quad (4)$$

and the value is independent of the choice of the center [11]. For convenience, we denote $B_P(\delta)$ and $B_Q(\delta)$ by $B(\delta)$ without distinguishing them. Now let $\theta_1, \dots, \theta_p$ be the principal angles between $P \in \mathcal{G}_{n,p}(\mathbb{L})$ and $Q \in \mathcal{G}_{n,q}(\mathbb{L})$, and $s_i = \sin^2 \theta_i$. The volume of a metric ball $B(\delta)$ is given by

$$\mu(B(\delta)) = \int_{0 \leq s_i^2 \leq 1, \sum_{i=1}^p s_i^2 \leq \delta} \dots \int d\mu_{\mathbf{s}}, \quad (5)$$

where the differential form $d\mu_{\mathbf{s}}$ is the joint density of s_i^2 's [7], [12]. If $p \leq q \leq \frac{n}{2}$, $d\mu_{\mathbf{s}}$ is explicitly given by

$$d\mu_{\mathbf{s}} = v_{n,p,q,\beta} |\Delta_p(\mathbf{s}^2)|^\beta \prod_{i=1}^p \left((s_i^2)^{\frac{\beta}{2}(n-p-q+1)-1} (1-s_i^2)^{\frac{\beta}{2}(q-p+1)-1} \right) ds_i^2,$$

where

$$|\Delta_p(\mathbf{s}^2)| = \prod_{1 \leq i < j \leq p} |s_i^2 - s_j^2|, \quad \beta = \begin{cases} 1 & \text{if } \mathbb{L} = \mathbb{R} \\ 2 & \text{if } \mathbb{L} = \mathbb{C} \end{cases}, \quad (6)$$

$v_{n,p,q,\beta} =$

$$\prod_{i=1}^p \frac{\Gamma\left(1 + \frac{\beta}{2}\right) \Gamma\left(\frac{\beta}{2}(n-i+1)\right)}{\Gamma\left(\frac{\beta}{2}i+1\right) \Gamma\left(\frac{\beta}{2}(n-q-i+1)\right) \Gamma\left(\frac{\beta}{2}(q-i+1)\right)}.$$

The evaluation of this multivariate integral (5) is generally difficult.

Our main result is that as $n, p, q, \delta \rightarrow \infty$ with $\frac{n}{n} \rightarrow \bar{p} \in (0, 1)$, $\frac{q}{n} \rightarrow \bar{q} \in (0, 1)$ and $\frac{\delta}{n} \rightarrow \bar{\delta} \in \mathbb{R}^+$, the volume of a metric ball $B(\delta)$ satisfies

$$\frac{1}{n^2} \log \mu(B(\delta)) \rightarrow -V$$

for a computable $V \geq 0$ (see Theorem 3 for details). Towards this result, an essential ingredient called the equilibrium measure is introduced in Section III-A. Then the volume growth formula is derived in Section III-B.

A. The Equilibrium Measure

We introduce the equilibrium measure related to the volume calculation as follows. For given $\alpha \geq 0$ and $\gamma \geq 0$, define

$$k(t, \tau) := \log |t - \tau|^{-1} + \frac{1}{2}V(t) + \frac{1}{2}V(\tau),$$

where

$$V(t) := \alpha \log t + \gamma \log(1 - t).$$

Let

$$\mathcal{M}^1([0, 1]) := \{ \mu \in \text{Borel measures on } [0, 1] : \mu([0, 1]) = 1 \}.$$

For any measure $\mu \in \mathcal{M}^1([0, 1])$, define

$$I(\mu) := \int \int k(t, \tau) \cdot d\mu(t) d\mu(\tau). \quad (7)$$

Now for a constant $\bar{\delta}' > 0$, we define the equilibrium measure $\mu_{\bar{\delta}'}$ $\in \mathcal{M}^1([0, 1])$ as the unique measure such that

$$I(\mu_{\bar{\delta}'}) = \inf_{\mu \in \mathcal{M}^1([0, 1]), \int t \cdot d\mu(t) \leq \bar{\delta}'} I(\mu). \quad (8)$$

The existence and uniqueness of $\mu_{\bar{\delta}'}$ is verified by the following lemma.

Lemma 1: For any $\bar{\delta}' > 0$, there exists a unique $\mu_{\bar{\delta}'}$ $\in \mathcal{M}^1([0, 1])$ such that

$$\inf_{\mu \in \mathcal{M}^1([0, 1]), \int t \cdot d\mu(t) \leq \bar{\delta}'} I(\mu) = I(\mu_{\bar{\delta}'}).$$

The proof of this Lemma follows that in [13, Chapter 6.2].

The physical meaning of the equilibrium measure can be described as follows. Randomly draw planes $P \in \mathcal{G}_{n,p}(\mathbb{L})$ and $Q \in \mathcal{G}_{n,q}(\mathbb{L})$ according to the isotropic measure. Let P^\perp and Q^\perp be the orthogonal complement planes of P and Q respectively. Define P' and Q' such that

$$\begin{cases} P' = P, Q' = Q & \text{if } p \leq q \leq \frac{n}{2} \\ P' = Q^\perp, Q' = P^\perp & \text{if } \frac{n}{2} \leq p \leq q \\ P' = P, Q' = Q^\perp & \text{if } p \leq \frac{n}{2} \leq q \text{ and } p + q \leq n \\ P' = Q^\perp, Q' = P & \text{if } p \leq \frac{n}{2} \leq q \text{ and } p + q > n \end{cases}.$$

Let $\theta'_1, \dots, \theta'_{p'}$ be the principal angles between P' and Q' , where p' is the dimension of the plane P' . Let

$$t_i = \begin{cases} \cos^2 \theta'_i & \text{if } p \leq \frac{n}{2} \leq q \\ \sin^2 \theta'_i & \text{otherwise} \end{cases}.$$

Define the conditional measure of t_i 's by

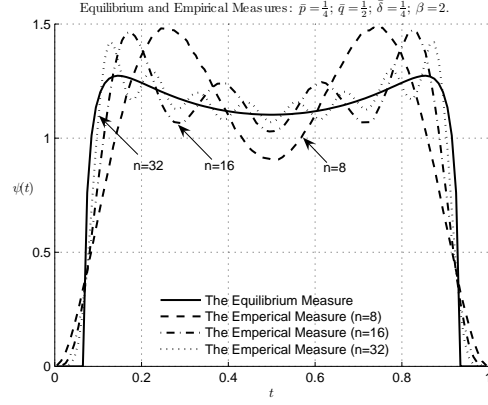
$$F(t|\delta) := \mathbb{E} \left\{ \frac{|\{i : t_i \leq t\}|}{p'} \mid d_c^2(P, Q) \leq \delta \right\}.$$

Now we set

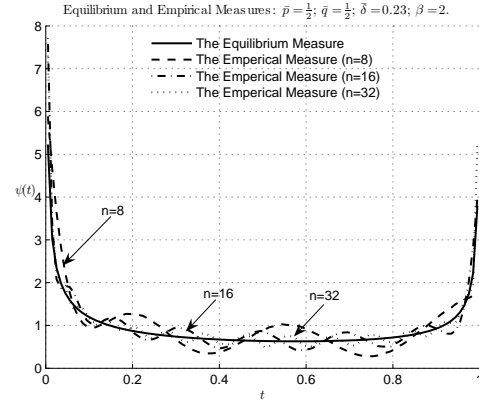
$$\begin{cases} \alpha = \frac{1-\bar{p}-\bar{q}}{\bar{p}}, \gamma = \frac{\bar{q}-\bar{p}}{\bar{p}}, \bar{\delta}' = \frac{1}{\bar{p}}\bar{\delta} & \text{if } \bar{p} + \bar{q} \leq 1 \\ \alpha = \frac{\bar{p}+\bar{q}-1}{1-\bar{q}}, \gamma = \frac{\bar{q}-\bar{p}}{1-\bar{q}}, \bar{\delta}' = \frac{1}{1-\bar{q}}\bar{\delta} & \text{if } \bar{p} + \bar{q} > 1 \end{cases}. \quad (9)$$

Then as $n, p, q, \delta \rightarrow \infty$ with constant ratios \bar{p}, \bar{q} and $\bar{\delta}$,

$$F(t|\delta) \rightarrow \mu_{\bar{\delta}'},$$



(a) An example with $\bar{\delta}' \geq \frac{1+\alpha}{2+\alpha+\gamma}$



(b) An example with $\bar{\delta}' < \frac{1+\alpha}{2+\alpha+\gamma}$

Figure 1. Convergence of the empirical measure

that is, for any continuous function g on $[0, 1]$ and Borel set $B \subset [0, 1]$,

$$\int_B g(t) \cdot dF(t|\delta) \rightarrow \int_B g(t) \cdot d\mu_{\bar{\delta}'}(t).$$

Fig. 1 exemplifies this convergence. The differential form of the equilibrium measure $\mu_{\bar{\delta}'}$ is evaluated according to Theorem 1. Two cases where $\bar{\delta}' \geq \frac{1+\alpha}{2+\alpha+\gamma}$ and $\bar{\delta}' < \frac{1+\alpha}{2+\alpha+\gamma}$ are presented.

Theorem 1 gives the explicit form of the equilibrium measure.

Theorem 1: For any $\alpha \geq 0$, $\gamma \geq 0$ and $\bar{\delta}' > 0$, the equilibrium measure $\mu_{\bar{\delta}'}$ is given by

$$\begin{aligned} \psi(t) := \frac{d\mu_{\bar{\delta}'}(t)}{dt} &= \frac{\sqrt{(t-a)(b-t)}}{2\pi} \\ &\cdot \left(\frac{\alpha}{\sqrt{ab}} \frac{1}{t} + \frac{\gamma}{\sqrt{(1-a)(1-b)}} \frac{1}{1-t} \right) \cdot \chi_{[a,b]}(t), \end{aligned} \quad (10)$$

where $\chi_{[a,b]}(t)$ is the indication function on $[a, b]$, $0 \leq a < b \leq 1$,

$$\begin{cases} a = 0 & \text{if } \alpha = 0 \\ a > 0 & \text{if } \alpha > 0 \end{cases},$$

$$\begin{cases} b = 1 & \text{if } \beta = 0 \\ b < 1 & \text{if } \beta > 0 \end{cases},$$

and

$$\begin{cases} 0 = \frac{\alpha}{\sqrt{ab}} - \frac{\gamma}{\sqrt{(1-a)(1-b)}} + \lambda \\ 2 + \alpha + \gamma = \frac{\gamma}{\sqrt{(1-a)(1-b)}} - \lambda \frac{a+b}{2} \\ 2 \min \left(\bar{\delta}', \frac{1+\alpha}{2+\alpha+\gamma} \right) = \frac{\gamma}{\sqrt{(1-a)(1-b)}} \frac{2-(a+b)}{2} \\ -\frac{\lambda}{8} (a-b)^2 - \gamma \end{cases}, \quad (11)$$

for some $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \lambda = 0 & \text{if } \bar{\delta}' \geq \frac{1+\alpha}{2+\alpha+\gamma} \\ \lambda \neq 0 & \text{if } \bar{\delta}' < \frac{1+\alpha}{2+\alpha+\gamma} \end{cases}.$$

The proof is omitted due to the space limitation.

The following examples give the equilibrium measure for some special cases.

Example 1: Let $\gamma = 0$ ($\bar{q} = \bar{p}$). Then $b = 1$ and

$$\psi(t) = \begin{cases} \frac{1}{2\pi} \sqrt{\frac{t-a}{1-t}} \left(\frac{\alpha}{\sqrt{a}} \frac{1}{t} + \lambda \right) & \text{if } \bar{\delta}' < \frac{1+\alpha}{2+\alpha} \\ \frac{2+\alpha}{2\pi} \frac{1}{t} \sqrt{\frac{t-(\frac{\alpha}{2+\alpha})^2}{1-t}} & \text{if } \bar{\delta}' \geq \frac{1+\alpha}{2+\alpha} \end{cases},$$

where $a \in (0, 1)$ and $\lambda \in \mathbb{R}$ are the solutions of (11).

Example 2: Let $\alpha = \gamma = 0$ ($\bar{p} = \bar{q} = \frac{1}{2}$). Then $a = 0$, $b = 1$ and

$$\psi(t) = \begin{cases} \frac{\sqrt{t(1-t)}}{2\pi} \left(\frac{6-8\bar{\delta}'}{t} + \frac{8\bar{\delta}'-2}{1-t} \right) & \text{if } \bar{\delta}' < \frac{1}{2} \\ \frac{1}{\pi\sqrt{t(1-t)}} & \text{if } \bar{\delta}' \geq \frac{1}{2} \end{cases}.$$

The closed form to compute $I(\mu_{\bar{\delta}'})$ is also obtained.

Theorem 2: Let a , b , $\frac{\alpha}{\sqrt{ab}}$, $\frac{\gamma}{\sqrt{(1-a)(1-b)}}$ and λ be in

Theorem 1. Let $c_1 = \left(\frac{\sqrt{a}+\sqrt{b}}{2} \right)^2$, $c_2 = \left(\frac{\sqrt{1-a}+\sqrt{1-b}}{2} \right)^2$, $\beta_1 = \left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}} \right)^2$, $\beta_2 = \left(\frac{\sqrt{1-a}-\sqrt{1-b}}{\sqrt{1-a}+\sqrt{1-b}} \right)^2$, $s_1 = \frac{\alpha}{\sqrt{ab}}$ and $s_2 = \frac{\gamma}{\sqrt{(1-a)(1-b)}}$. Then

$$I(\mu_{\bar{\delta}'}) = \frac{1}{2} \left(l_0 + \lambda \min \left(\bar{\delta}', \frac{1+\alpha}{2+\alpha+\gamma} \right) - \alpha l_1 - \gamma l_2 \right),$$

where

$$\begin{aligned} l_0 &= -2s_1c_1\beta_1 \left[\log(c_1\beta_1^{\frac{1}{2}}) + \frac{\beta_1-1}{\beta_1} \log(1+\beta_1^{\frac{1}{2}}) + \beta_1^{-\frac{1}{2}} \right] \\ &\quad - 2s_2c_2\beta_2 \left[\log(c_2\beta_2^{\frac{1}{2}}) + \frac{\beta_2-1}{\beta_2} \log(1-\beta_2^{\frac{1}{2}}) - \beta_2^{-\frac{1}{2}} \right] \\ &\quad - \alpha \log b - \gamma \log(1-b) - \lambda b, \end{aligned}$$

$$\begin{aligned} l_1 &= s_2c_2\beta_2 \left[\log(1-c_2-c_1\beta_1) + \frac{1}{\beta_2} \log(1-c_2\beta_2-c_1\beta_1) \right] \\ &\quad + s_2c_1\beta_1 + s_1c_1\beta_1 \left[\log c_1 + \frac{\beta_1-1}{\beta_1} \log(1-\beta_1) - 1 \right], \end{aligned}$$

and

$$\begin{aligned} l_2 &= s_1c_1\beta_1 \left[\log(1-c_1-c_2\beta_2) + \frac{1}{\beta_1} \log(1-c_2\beta_2-c_1\beta_1) \right] \\ &\quad + s_1c_2\beta_2 + s_2c_2\beta_2 \left[\log c_2 + \frac{\beta_2-1}{\beta_2} \log(1-\beta_2) - 1 \right]. \end{aligned}$$

The proof is omitted due to the space limitation.

B. The Volume Growth Formula

Based on the equilibrium measure, we reach our volume growth formula.

Theorem 3: Let $n, p, q, \delta \rightarrow \infty$ with $\frac{p}{n} \rightarrow \bar{p} \in (0, 1)$, $\frac{q}{n} \rightarrow \bar{q} \in (0, 1)$ and $\frac{\delta}{n} \rightarrow \bar{\delta} > 0$. Then

$$\lim_{(n,p,q,\delta) \rightarrow \infty} \frac{1}{n^2} \log \mu(B(\delta)) = -V_{\bar{p}, \bar{q}, \bar{\delta}, \beta}, \quad (12)$$

where β is defined in (6),

$$V_{\bar{p}, \bar{q}, \bar{\delta}, \beta} := \begin{cases} \frac{\beta}{4} (\bar{p}')^2 (2I(\mu_{\bar{\delta}'}) - \bar{v}_{\bar{p}', \bar{q}'}) & \text{if } \bar{\delta}' \leq \bar{p}'(1-\bar{q}') \\ 0 & \text{if } \bar{\delta}' > \bar{p}'(1-\bar{q}') \end{cases}, \quad (13)$$

$$\begin{cases} \bar{p}' = \bar{p}, & \bar{q}' = \bar{q}, & \bar{\delta}' = \frac{\bar{\delta}}{\bar{p}} & \text{if } \bar{p} + \bar{q} \leq 1 \\ \bar{p}' = 1 - \bar{q}, & \bar{q}' = \bar{p}, & \bar{\delta}' = \frac{\bar{\delta}}{1-\bar{q}} & \text{if } \bar{p} + \bar{q} > 1 \end{cases},$$

$\mu_{\bar{\delta}'}$ is the corresponding equilibrium measure defined in (8), $I(\mu_{\bar{\delta}'})$ is given by Theorem 2,

$$\bar{v}_{\bar{p}', \bar{q}'} := J\left(\frac{1}{\bar{p}'}\right) - J\left(\frac{1-\bar{q}'}{\bar{p}'}\right) - J\left(\frac{\bar{q}'}{\bar{p}'}\right), \quad (14)$$

and

$$J(t) := t^2 \log t - (t-1)^2 \log(t-1) \quad \text{for } t \geq 1.$$

The proof of Theorem 3 is sketched as follows. There are totally four cases into consideration: the $p \leq q \leq \frac{n}{2}$ case, the $\frac{n}{2} \leq p \leq q$ case, the case where $p \leq \frac{n}{2} \leq q$ and $p+q \leq n$, and the case where $p \leq \frac{n}{2} \leq q$ and $p+q \geq n$. We omit the proofs for the last three cases because they are similar to that for the first case under minor changes. Recall the multivariate integral representation of volume in (5). Note that $v_{n,p,q,\beta}$ has an explicit formula in terms of Gamma functions and so an application of Stirling's formula gives its asymptotic behavior.

Theorem 4: As $n, p, q \rightarrow \infty$ with constant ratios $\bar{p}, \bar{q} \in (0, 1)$,

$$\frac{1}{p^2} \log v_{n,p,q,\beta} \rightarrow \frac{\beta}{4} \bar{v}_{\bar{p}, \bar{q}},$$

where β and $\bar{v}_{\bar{p}, \bar{q}}$ are defined in (6) and (14) respectively.

The limit of the other part in the volume representation (5) can also be calculated from the equilibrium measure.

Theorem 5: As $n, p, q, \delta \rightarrow \infty$ with constant ratios $\bar{p}, \bar{q} \in (0, 1)$ and $\bar{\delta} > 0$,

$$\frac{1}{p^2} \log \int \cdots \int_{0 \leq s_i^2 \leq 1, \sum s_i^2 \leq \delta} \frac{d\mu_s}{v_{n,p,q,\beta}} \rightarrow -\frac{\beta}{2} I\left(\mu_{\frac{1}{\bar{p}}\bar{\delta}}\right),$$

where β , $I(\cdot)$ and $\mu_{\frac{1}{\bar{p}}\bar{\delta}}$ are given in (6), (7) and (8) respectively.

The proof of this theorem is based on large deviation techniques and omitted here due to the space limitation.

Combining Theorem 4 and 5 proves Theorem 3.

Fig 2 illustrates how accurate is to apply our volume growth formula (12) (valid when $n \rightarrow \infty$) to Grassmann manifolds with finite n . The solid line is from our volume growth formula and the dotted lines are for Grassmann manifolds with finite n . From Fig 2, the curve for $n = 16$ is close to the volume growth formula, and the difference between the $n = 32$ case and our formula is almost invisible.

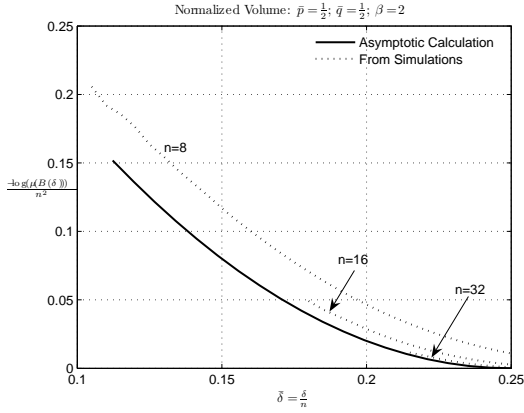


Figure 2. Convergence of the volume of a metric ball

IV. THE RATE DISTORTION TRADEOFF

This section quantifies the rate distortion tradeoff for the unequal dimensional quantization problem.

Recall the distortion rate function defined in (2). It is now quantified based on the volume growth formula (12).

Theorem 6: Let $n, p, q, K \rightarrow \infty$ with $\frac{p}{n} \rightarrow \bar{p} \in (0, 1)$, $\frac{q}{n} \rightarrow \bar{q} \in (0, 1)$ and $\frac{\log K}{n^2} \rightarrow \bar{r} \in \mathbb{R}^+$. Let $0 \leq \bar{\delta}_r \leq \bar{p}(1 - \bar{q})$ be the unique solution of $V_{\bar{p}, \bar{q}, \bar{\delta}_r, \beta} = \bar{r}$, where $V_{\bar{p}, \bar{q}, \bar{\delta}_r, \beta}$ is given in (13). Then

$$\lim_{(n, p, q, K) \rightarrow \infty} \frac{1}{n} D^*(K) = \bar{\delta}_r.$$

The proof of this theorem follows the same idea as that in [9], [10]. We first construct a lower bound on the distortion rate function by a sphere covering argument. Then an upper bound is obtained by calculating the average distortion of random codes. Since the upper bound asymptotically meets the lower bound, the asymptotic distortion rate function is precisely quantified.

As the dual part of the distortion rate tradeoff, the rate distortion function is now quantified.

Corollary 1: For required distortion D , as $n, p, q, D \rightarrow \infty$ with $\frac{p}{n} \rightarrow \bar{p} \in (0, 1)$, $\frac{q}{n} \rightarrow \bar{q} \in (0, 1)$ and $\frac{D}{n} \rightarrow \bar{D} > 0$,

$$\lim_{(n, p, q, D) \rightarrow \infty} \frac{1}{n^2} \log K^*(D) = V_{\bar{p}, \bar{q}, \bar{D}, \beta},$$

where $V_{\bar{p}, \bar{q}, \bar{D}, \beta}$ is given in (13).

Remark 1: According to Corollary 1, the quantization rate required for a given normalized distortion \bar{D} increases linearly with n^2 as $n \rightarrow \infty$. Note that the real dimensions of the Grassmann manifold $\mathcal{G}_{n, p}(\mathbb{L})$ are $\beta p(n - p) \propto n^2$. Corollary 1 is consistent with the intuition that for a given \bar{D} , the minimum required quantization rate should be scaled by the number of real dimensions.

Remark 2: The derived tradeoff is different from our previous results in [8]–[10]. The results there are valid only when $D \leq 1$ ($\bar{D} = \frac{D}{n} \rightarrow 0$ and $V_{\bar{p}, \bar{q}, \bar{D}, \beta} \rightarrow \infty$ in the language of this paper). We refer to the quantization there as high rate quantization.

While the asymptotic rate distortion tradeoff is precisely quantified, the next question could be how to achieve it. As

is often the case, *random codes are asymptotically optimal in probability.*

Theorem 7: Let $\mathcal{C}_{\text{rand}}^{(n)} \subset \mathcal{G}_{n, q}(\mathbb{L})$ be a code randomly generated from the isotropic distribution and with size K . Suppose that $n, p, q, K \rightarrow \infty$ with $\frac{p}{n} \rightarrow \bar{p} \in (0, 1)$, $\frac{q}{n} \rightarrow \bar{q} \in (0, 1)$ and $\frac{\log K}{n^2} \rightarrow \bar{r}$. Let $0 \leq \bar{\delta}_r \leq \bar{p}(1 - \bar{q})$ be the unique solution of $V_{\bar{p}, \bar{q}, \bar{\delta}_r, \beta} = \bar{r}$ where $V_{\bar{p}, \bar{q}, \bar{\delta}_r, \beta}$ is given in (13). Then for $\forall \epsilon > 0$,

$$\lim_{(n, p, q, K) \rightarrow \infty} \Pr \left(\frac{1}{n} D(\mathcal{C}_{\text{rand}}^{(n)}) > \bar{\delta}_r + \epsilon \right) = 0$$

The proof is omitted due to the space limitation.

V. CONCLUSION

This paper considers the unequal dimensional quantization on the Grassmann manifold. A volume growth formula is derived as $n, p, q, \delta \rightarrow \infty$ linearly. Based on this formula, the asymptotic rate distortion tradeoff is precisely quantified. The random codes are proved to be asymptotically optimal in probability.

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