Fluctuations in the Thermodynamic Limit of Focussing Cubic Schrödinger

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Abstract The invariant ensemble for nonlinear Schrödinger with periodic conditions was introduced by Lebowitz-Rose-Speer [5]. In the focussing cubic case with the circle of perimeter L the ensemble is described as follows. The field (Q,P) is a two-dimensional Circular Brownian Motion subject to the Radon-Nykodym factor $\exp[\int_0^L (Q^2+P^2)^2]$ which is balanced by conditioning on the constant of motion $\int_0^L (Q^2+P^2) = LD$ for positive density D. The competition between the quartic-potential and the micro-canonical fiat resolves itself in vast local concentration causing the ensemble to collapse onto the unit measure on the zero path as $L \uparrow \infty$. An obvious question is then to scale the paths appropriately as L gets large in order to capture fluctuations away from this trivial thermodynamic limit. The present note studies a discrete caricature of the ensemble and proves that at a scaling of $\sqrt{L}$ and high density one has fluctuations resembling a White Noise.

Running title Fluctuations for NLS

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1 Introduction and Results

The focussing cubic Schrödinger equation considered on the circle of perimeter L has the Hamiltonian formalism,[2]

$$Q^\bullet = -P'' - (P^2+Q^2)P = \partial H/\partial P \quad \text{and} \quad P^\bullet = +Q'' + (P^2+Q^2)Q = -\partial H/\partial Q,$$

with

$$H = \frac{1}{2}\int_0^L [(P')^2 + (Q')^2]dx - \frac{1}{4}\int_0^L [P^2+Q^2]^2 dx.$$

Following the classical prescription of Gibbs, Lebowitz-Rose-Speer (LRS) [5] introduced the petit canonical ensemble as a candidate for an invariant measure of the flow:

$$e^{-H} d^\infty P d^\infty Q = e^{+(1/4)\int_0^L (P^2+Q^2)^2} \times \frac{e^{-(1/2)\int_0^L (P')^2}}{(2\pi 0+)^{\infty/2}} d^\infty P \times \frac{e^{-(1/2)\int_0^L (Q')^2}}{(2\pi 0+)^{\infty/2}} d^\infty Q. \tag{1}$$

The meaning of this formal object is easy to explain. The third factor signifies that Q is a "Circular Brownian Motion," *i.e.*, it is standard Brownian Motion

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[2]$\bullet$ signifies $\partial/\partial t$.

starting at $Q(0) = m$ conditioned to come back to m at $x = L$, this common value being distributed over the line according to the infinite measure $(2\pi L)^{-1/2} dm$. The second factor has the same meaning for P. The first part is just a density, having a proper sense because the Brownian path is continuous. Unfortunately, the ensemble as defined has infinite total mass. This is remedied by taking a micro-canonical viewpoint in which the measure is restricted by fixing the constant of motion $\int_0^L (P^2 + Q^2) = LD$ with positive density D. That is, we consider the probability measure on paths with partition function:[3]

$$\mathfrak{Z}_L = \int_{\mathbb{R}^2} \mathbf{E}_c \Big[e^{\frac{1}{4}\int_0^L [Q^2(x)+P^2(x)]^2 dx}, \int_0^L Q^2(x) + P^2(x) dx = LD, (Q,P)(L) = c \Big] dc.$$

Here $\mathbf{E}_\bullet$ is the mean of the planar Brownian Motion (Q, P) starting at $\bullet \in \mathbb{R}^2$. The fact that now $\mathfrak{Z}_L < \infty$ was first proved by LRS.

That the ensemble defined by $\mathfrak{Z}_L$ is indeed invariant under the cubic Schrödinger flow for any $L < \infty$ was proven independently by McKean [6] and Bourgain [2]. McKean also discussed the question of the thermodynamic ($L \uparrow \infty$) limit: in [7] a proof was put forward that the full limit does not exist. That is, depending on how the circle was taken to the whole line, the micro-canonical ensemble would give way to an infinity of Gibbs states. This was taken as a possible explanation for the differing numerical results of other authors. Simulations of LRS suggested a phase transition: the ensemble living near solitons/radiation for high/low values of D. The work of [3] appeared to run counter to this interpretation.

Unfortunately, [7] contains an error. In fact, not only does the thermodynamic limit exist, it is trivial: as $L \uparrow \infty$ the ensemble collapses onto the unit measure on the trivial path. This is spelled out in [10] where the free energy is computed:

$$\lim_{L\uparrow\infty} \frac{1}{L^3} \log \mathfrak{Z}_L = \sup_{\int_{-\infty}^{\infty} |f(x)|^2 dx = D} \Big\{ \frac{D}{4} \int_{-\infty}^{\infty} |f(x)|^4 dx - \frac{1}{2} \int_{-\infty}^{\infty} |f'(x)|^2 dx \Big\}; \qquad (2)$$

it is a positive and continuous function of D - no phase transition is seen on this level. For all D, the leading paths which contribute to (2) live near a single soliton of height L and width $1/L$. As $L \uparrow \infty$, it is this vast local concentration of the field that results in the collapse.

The obvious next question is to understand fluctuations away from this trivial limit. That is, for some rate γ_L tending to ∞ with L, one wants to determine a possible limit law for the field $(\gamma_L Q, \gamma_L P)$ under the micro-canonical measure. Unfortunately, such a computation in the present ensemble is beyond us. Therefore, in hope of shedding some light on the matter, here we introduce and study a discrete caricature of the cubic Schrödinger system.

[3]The peculiar but useful notation $(Q, P)(L) = c$ and the like indicate densities: $E[F(Q) = a, G(P) = b] = (\partial^2/\partial N \partial M) E[F(Q) \le N, G(P) \le M]|_{N=a,M=b}$.

The idea is simple: replace the Brownian Motion with a Gaussian Random Walk. Making things one-dimensional as well, we introduce the Hamiltonian:[4]

$$H_{L,\Delta}(Q) = -\frac{1}{4}\sum_{k=0}^{\lceil L/\Delta\rceil -1} Q_k^4 \Delta + \frac{1}{2}\sum_{k=0}^{\lceil L/\Delta\rceil -1}\left(\frac{Q_{k+1}-Q_k}{\Delta}\right)^2 \Delta, \tag{3}$$

for periodic fields Q, $Q_0 = Q_{\lceil L/\Delta\rceil}$, subject of course to the micro-canonical condition $\sum_{k=0}^{\lceil L/\Delta\rceil -1} Q_k^2\Delta = DL$. Finally, in order to best mimic the diffusion ensemble, the step size Δ is taken to depend on L as in $\Delta = 1/L$; the appropriateness of this choice is explained below.

Our object of study is then the model measure

$$d\mathbf{M}_L = \exp\left[-H_{L,\frac{1}{L}}(Q)\right]\left(\prod_{k=0}^{L^2-1} dQ_k\right)\Big|_{\left\{\sum_{k=0}^{L^2-1} Q_k^2 \frac{1}{L} = LD\right\}} : \tag{4}$$

a circular Gaussian random walk, weighted by a quartic potential and conditioned to remain on the sphere of radius $\sqrt{LD}$. As in the diffusion case, the ensemble $\mathbf{M}_L$ collapses onto $\delta(Q \equiv 0)$ as $L \uparrow \infty$. Again the paths concentrate near a soliton. Now however we are able to capture fluctuations away from the trivial thermodynamic limit, at least for large enough density. The main result of the paper is the following.

Theorem 1 For D sufficiently large, the limiting distribution of the scaled field $Q \to \sqrt{L}Q$ under $\mathbf{M}_L$ is a White Noise. More precisely, for any finite collection of positions $x_0 < x_1 < \cdots x_m$ we have: with ρ_D a positive constant to be defined below,

$$\lim_{L\uparrow\infty} \mathbf{M}_L\left[\sqrt{L}Q_{Lx_0} = a_0, \sqrt{L}Q_{Lx_1} = a_1, \ldots, \sqrt{L}Q_{Lx_m} = a_m\right] = \prod_{k=0}^{m} \frac{\exp\left(-\rho_D a_k^2\right)}{\sqrt{\pi/\rho_D}}$$

in the sense of weak convergence of measures. This follows from computing the following limiting joint density

$$\lim_{L\uparrow\infty} \mathbf{M}_L\left[\sqrt{L}Q_0 = a, \sqrt{L}Q_{Lx} = b, L(\sum_{k=1}^{Lx-1} Q_k^2 \frac{1}{L}) = I\right]$$
$$= \frac{\exp\left(-\rho_D a^2\right)}{\sqrt{\pi/\rho_D}} \times \frac{\exp\left(-\rho_D b^2\right)}{\sqrt{\pi/\rho_D}} \times \delta\left(I - \frac{x}{2\rho_D}\right), \tag{5}$$

which should be interpreted as the density of $[Q_\infty(0), Q_\infty(x), \int_0^x Q^2(x')dx']$.

[4]From here on we will drop the usage of $\lceil$ and $\rceil$. It will be clear from the context when we are running through integers.

That we have the result for large D only stems from a missing uniqueness statement in a discrete variational problem connected to the Hamiltonian (3). While we believe the above obtains at all D, it must be said up front that the possibility of a phase transition still exists.

The rest of the paper is devoted to proving Theorem 1. We begin in Section 2 with various technicalities. First, the measure $\mathbf{M}_L$ is discussed in greater detail and an auxiliary measure through which $\mathbf{M}_L$ is better studied is introduced. We also state a Lemma on the optimal configurations for maximizing (3) on the sphere as these clearly govern the behavior of $\lim_{L\uparrow\infty} \mathbf{M}_L$. Section 3 computes the limit law of the scaled marginal $\sqrt{L} \times Q_0$. The computation of (5) is completed in Section 4.

2 Preliminaries

2.1 The measure $\mathbf{M}_L$

Bringing in $\mathbf{E}_\bullet$ = the expectation starting at $\bullet \in \mathbb{R}$ of a centered Gaussian random walk of step size $\Delta = 1/L$ and mean square $\Delta^{-1} = L$, our discrete ensemble $\mathbf{M}_L$ is expressed more concretely through its partition function

$$\mathbf{Z}_L = \int_{-\sqrt{LD}}^{\sqrt{LD}} \mathbf{E}_c \Big[e^{\frac{1}{4L} \sum_0^{L^2-1} Q_k^4}, \sum_{k=0}^{L^2-1} Q_k^2 \frac{1}{L} = LD, Q_{L^2} = c \Big] dc.$$

Note that $\mathbf{M}_L$ inherits the rotation invariance of the underlying walk. We also introduce the density

$$p(x, a, b, I) = \mathbf{E}_a \Big[e^{\frac{1}{4L} \sum_0^{Lx-1} Q_k^4}, \sum_{k=0}^{Lx-1} Q_k^2 \frac{1}{L} = I, Q_{Lx} = b \Big], \tag{6}$$

through which many quantities of interest for $\mathbf{M}_L$ can be expressed: *e.g.*, $\mathbf{Z}_L = \int p(L, c, c, LD) dc$ and the marginal density is

$$\mathbf{M}_L[a] = \mathbf{M}_L \Big[Q_0 = a \Big] = \mathbf{Z}_L^{-1} p(L, a, a, LD). \tag{7}$$

The arguments of [10] can be adapted to show that $\mathbf{M}_L[a]da \to \delta_0$, from which the collapse is evident by rotation invariance. That is not dwelled on here. The present goal is to understand *scaled* quantities: $L^{-1/2}\mathbf{M}_L[L^{-1/2}a]$ and the like.

Before moving on, a word is in order as to our choice $\Delta = 1/L$. What makes the analysis of the diffusion or $\mathfrak{Z}_L$ ensemble challenging is the competition between the unbounded quartic potential held down mean-square which keeps the measure finite but leads to paths that want to concentrate locally and the gradient square term of the (Brownian) energy which likes things smooth. Now, ignoring for a second the periodicity and uninteresting constant multipliers, in the present discrete setup the total mass $\mathbf{Z_L}$ is seen to have the form:

$$\mathbf{Z}_L \simeq \int_{\sum_0^{L^2-1} \sigma_i^2 = LD} \exp\Big[\frac{1}{4}\frac{1}{L}\sum_{i=0}^{L^2-1}\sigma_i^4 - \frac{L}{2}\sum_{i=0}^{L^2-2}(\sigma_{i+1}-\sigma_i)^2\Big] d\sigma \tag{8}$$

$$= c_{L,D}\int_{S_1^{L^2-1}} \exp\Big[L^3 D\Big\{\frac{D}{4}\sum_{i=0}^{L^2-1}\sigma_i^4 + \sum_{i=0}^{L^2-2}\sigma_i\sigma_{i+1}\Big\} + L^{3/2}\sqrt{D}(\sigma_1 + \sigma_{L^2-1})\Big] d\sigma.$$

Thus the competition between $\sum \sigma_i^4$ (favoring a soliton) and $\sum \sigma_i\sigma_{i+1}$ (favoring radiation) is present here. Also, it is relatively easy to see that $\lim_{L\uparrow\infty} L^{-3}\log \mathbf{Z}_L = \max\{(1/4)\sum_{-\infty}^{\infty}\sigma_k^4 + \sum_{-\infty}^{\infty}(\sigma_k - \sigma_{k+1})^2 \text{ on } \sum_{-\infty}^{\infty}\sigma^2 = D\}$ in which you have the same rate with free energy analogous to that of $\mathfrak{Z}_L$. Finally, as the diffusion ensemble concentrates near a single soliton of width $\simeq 1/L$, one should take $\Delta \downarrow 0$ at least as fast in hopes of "sampling" the diffusion at the correct scale.

2.2 The auxiliary measure

Examining the exponent in (8), it is clear that in order to understand $\mathbf{M}_L$ one must investigate the quartic form[5]

$$H_{L^2,D}(\sigma) = \frac{D}{4}\sum_{i=-L^2/2}^{L^2/2}\sigma_i^4 + \sum_{i=-L^2/2}^{L^2/2-1}\sigma_i\sigma_{i+1} \tag{9}$$

maximized over $\sum_{i=-L^2/2}^{L^2/2}\sigma_i^2 = 1$. The Lagrange multiplier for this problem will also be important below. It is

$$\lambda_{L^2,D}(\sigma) = D\sum_{-L^2/2}^{L^2/2}\sigma_i^4 + 2\sum_{-L^2/2}^{L^2/2}\sigma_i\sigma_{i+1} = 2H_{L^2,D}(\sigma) + \frac{D}{2}\sum_{-L^2/2}^{L^2/2}\sigma_i^4. \tag{10}$$

Next, the study of the scaled $\mathbf{M}_L$ ensemble is transfered to that of the auxiliary measure μ_{L^2} on the L^2 dimensional sphere

$$d\mu_{L^2}\Big(\sigma_{-L^2/2}, \sigma_{-L^2/2+1}, \ldots, \sigma_{L^2/2}\Big)$$

$$= \frac{1}{z_{L^2}}\exp\Big[DL^3 H_{L^2,D}(\sigma)\Big]\delta\Big(\sum_{-L^2/2}^{L^2/2}\sigma_i^2 = 1\Big) d\sigma_{-L^2/2}\ldots d\sigma_{L^2/2} \tag{11}$$

with its own private partition function z_{L^2}. Much of the proof involves re-expressing $\mathbf{M}_L$ averages in terms of $\mathbf{E}^{\mu_{L^2}}$ averages.

[5]We will usually suppress the dependence of H_{L^2} and λ_{L^2} on D. Note also the re-indexing.

The parameter ρ_D appearing as the limiting mean-square in Theorem 1 may now be defined: provided the limit exists (which is proved for $D \gg 1$),

$$\rho_D = \sqrt{\lim_{L\uparrow\infty} \mathbf{E}^{\mu_L^2}[\lambda^2_{L^2,D}(\sigma)/4] - 1}.$$

This reflects the structure of the Hamiltonian and has an interesting behavior as a function of the density D: $\rho_D \uparrow \infty$ with D and $\rho_0 = 0$. So, believing Theorem 1 to hold at all D one would have that fluctuations away from the collapse are increasingly heavy-tailed as $D \downarrow 0$. One explanation of the numerical experiments may be that while solitons persist, the relatively high fluctuations at small D obscures this from view.

2.3 The discrete Hamiltonian

As mentioned, the variational problem inherent in (9) - maximizing H_{L^2} on the sphere - plays a central role in the sequel. The next Lemma states all that we know; the proof is found in the Appendix.

Lemma 1 Denoting

$$m_{L^2,D} = \max_{\sum_{-L^2/2}^{L^2/2}\sigma_i^2=1} H_{L^2,D}(\sigma) \quad \text{and} \quad m_{\infty,D} = \sup_{\sigma\in\ell^2} H_{\infty,D}(\sigma),$$

we have concentration of μ_{L^2} as in

$$\mu_{L^2}\Big(H_{L^2,D}(\sigma) < m_{\infty,D} - \varepsilon\Big) \leq c_1\varepsilon^{-L^2} e^{-c_2 L^3 \varepsilon} \tag{12}$$

as well as the following.

(a) For $L \uparrow \infty$ and any $D > 0$, $m_{L^2,D} > 1$. More precisely we know $m_{L^2,D} \geq 1 + D^2/32$ for $D \ll 1$ while $m_{L^2,D} \geq D/4 + 7/4D - \mathrm{O}(D^{-3})$ for $D \gg 1$.

(b) For all $D > 0$ the maximizing σ resembles a soliton: it is largest at $k = 0$ and is increasing/decreasing to the left/right. In fact, the maximizer decays exponentially far out, as in $|\sigma_k| \leq c_1 e^{-c_2|k|}$ for $|k| \geq M$ for some large M with c_1 and c_2 depending only on D. For large D the decay is sharper as in $\sigma_k \sim D^{-|k|}$. Furthermore, the maximum $m_{\infty,D}$ is attained for all $D > 0$.

(c) Modulo the obvious reflection, the maximizer of H is unique for all sufficiently large values of D. It follows that $\lambda_{L^2,D}$ converges to a constant under μ_{L^2}.

Remark The real barrier to having Theorem 1 for all densities is the fact that we have the above uniqueness statement (part **(c)**) for large D only. Indeed, the matter of uniqueness is much harder than one would first guess.

2.4 Outline of the computation

We will start with one-dimensional marginal

$$\mathbf{M}_L[a] = Z_L^{-1} p(L,a,a,LD) = \mathbf{M}_L[0]\frac{p(L,a,a,LD)}{p(L,0,0,LD)}.$$

Now the density p has an explicit expression as a spherical integral: to wit,

$$p(L,a,a,LD) = (L/2\pi)^{L^2/2}\ (DL^2-a^2)^{\frac{L^2-2}{2}}\exp(-L^3D)\exp(a^4/4L)$$
$$\times\int_{\sum_{k=1}^{L^2-1}\sigma_k^2=1}\exp\Big[\frac{1}{4L}(DL^2-a^2)^2\sum_{k=1}^{L^2-1}\sigma_k^4 + L(DL^2-a^2)\sum_{k=1}^{L^2-2}\sigma_k\sigma_{k+1}\Big]d\sigma.$$

This allows us to re-write $\mathbf{M}_L[a]$ in terms of the measure μ_{L^2} defined in (11):

$$\mathbf{M}_L\Big[Q_0\in da\Big] = \mathbf{M}_L[0](1-\frac{a^2}{DL^2})^{\frac{L^2-3}{2}}\exp(a^4/4L) \tag{13}$$
$$\times\mathbf{E}^{\mu_{L^2}}\Big[\exp\Big\{-L\lambda_{L^2}(\sigma)\frac{a^2}{2}+La\sqrt{L^2D-a^2}(\sigma_{-L^2/2}+\sigma_{L^2/2})+\frac{a^4}{4L}\sum\sigma^4\Big\}\Big]da.$$

Next, as tightness obtains ($\mathbf{M}_L[Q_0^2]=D$) the scaled marginal (take a into $\sqrt{L}a$) may be shown to satisfy: for a bounded and L large,

$$\mathbf{M}_L\Big[\sqrt{L}Q_0\in da\Big] = \frac{\mathbf{M}_L[0]}{\sqrt{L}}\mathbf{E}^{\mu_{L^2}}\Big[e^{\{-\lambda_{L^2}(\sigma)a^2/2+L^{3/2}\sqrt{D}(\sigma_{-L^2/2}+\sigma_{L^2/2})a\}}\Big]da \tag{14}$$

up multiplicative errors $1+\mathrm{O}(1/L)$ on the right hand side. Now, μ_{L^2} concentrates sharply at the maximizers of H_{L^2} and Lemma 1 (**b**) shows that the tail variables $(\sigma_{-L^2/2},\sigma_{L^2/2})$ are exponentially small in L at $H=max$. Again by the concentration of μ_{L^2} it is natural to hope that λ_{L^2} should be roughly constant for $L\uparrow\infty$, and thus that (14) should settle down to a centered Gaussian with mean-square one over λ_∞; the tail variables being unimportant.

However, this reasoning is just *wrong*. The tail variables *do* figure in: $\sigma_{L^2/2}$ exhibits enough fluctuation (away from zero) that $\mathbf{E}^{\mu_{L^2}}[\exp(L^{3/2}\sqrt{D}a\sigma_{L^2/2})]$ is strictly-positive for $L\uparrow\infty$.[6] We will need the following, the proof of which is left to the Appendix.

Lemma 2 There exists some positive γ depending on D such that

$$\limsup_{L\uparrow\infty}\mathbf{E}^{\mu_L^2}\Big[\exp\big(L^3\gamma\sigma_{L^2/2}^2\big)\Big]<\infty.$$

For D sufficiently large this will also imply that $L^{-1/2}\mathbf{M}_L[0]=O(1)$ for $L\uparrow\infty$.

[6] The symmetry of H_{L^2} implies $\sigma_{-L^2/2}$ and $\sigma_{L^2/2}$ are equally distributed.

Given Lemma 2 we will show the pair $(L^{3/2}\sigma_{-L^2/2}, L^{3/2}\sigma_{L^2/2})$ has a Gaussian limit for $L \uparrow \infty$. This is the main argument needed to establish convergence of the scaled $\mathbf{M}_L$ marginal and occupies the next section. With the marginal in hand, we then turn (Section 4) to the computation of the limiting scaled joint density (5) central to Theorem 1. That too has an expression in term of the p's:

$$\mathbf{M}_L\Big[\sqrt{L}Q_0 = a, \sqrt{L}Q_{Lx} = b, \sum_{k=0}^{Lx-1}(\sqrt{L}Q_k)^2\frac{1}{L} = I\Big]$$
$$= \frac{1}{L^2\mathbf{Z_L}}p\big(x, \frac{a}{\sqrt{L}}, \frac{b}{\sqrt{L}}, \frac{I}{L} + \frac{a^2}{L^2}\big)p(L-x, \frac{b}{\sqrt{L}}, \frac{a}{\sqrt{L}}, DL - \frac{I}{L} - \frac{a^2}{L^2}). \qquad (15)$$

3 The scaled marginal distribution

Our objective here is to establish:

Proposition 1 Let D be large. Then for $L \uparrow \infty$ the law of the tail variables $(L^{3/2}\sqrt{D}\sigma_{-L^2/2}, L^{3/2}\sqrt{D}\sigma_{L^2/2})$ converges to a pair of independent centered Gaussians with variance $\lambda/2 - \sqrt{\lambda^2/4 - 1}$. Here $\lambda = \lambda_{D,\infty} > 2$. As an immediate corollary one gets the convergence in law of the scaled $\mathbf{M}_L$ marginal as in

$$\lim_{L\uparrow\infty}\mathbf{M}_L\Big[\sqrt{L}Q_0 \in da\Big] = \lim_{L\uparrow\infty}\frac{\mathbf{M}_L[0]}{\sqrt{L}}\mathbf{E}^{\mu_L^2}\Big[e^{-\frac{1}{2}\lambda_{L^2}(\sigma)a^2 + \sqrt{D}L^{3/2}(\sigma_{-L^2/2}+\sigma_{L^2/2})}\Big]$$
$$= \sqrt{\frac{\rho_D}{\pi}}\exp\left[-\rho_D a^2\right]da \qquad (16)$$

where $\rho_D = \sqrt{\lambda^2/4 - 1}$. Note this also pins down the convergence of $L^{-1/2}\mathbf{M}_L[0]$ for $D \gg 1$.

The proof of Proposition 1 requires the following two Lemmas. The strategy is to determine an integral equation satisfied by any limiting density function of the tail variables and then search for solutions.

Lemma 3 Under μ_{L^2}, the law of $L^{3/2}\sqrt{D}\sigma_{L^2/2}$ is tight for $L \uparrow \infty$ (assuming large D); any limiting density function f is even and solves

$$f(x) = f(0)\exp(-\lambda x^2/2)\int_{-\infty}^{\infty}\exp(xy)f(y)dy. \qquad (17)$$

Likewise, any limiting joint density

$$f(x, y) = \lim_{L'\uparrow\infty}\mu_{L^2}(L^{3/2}\sqrt{D}\sigma_{-L^2/2} = x, L^{3/2}\sqrt{D}\sigma_{L^2/2} = y)$$

satisfies

$$f(x,y) = f(0,0)\exp\left(-\lambda(x^2+y^2)/2\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(xz+yw\right)f(z,w)dzdw \quad (18)$$

with obvious symmetries $f(x,y) = f(-x,-y) = f(y,x)$.

One may check that $f_* \otimes f_*$ with $f_*(x) = \sqrt{\Lambda/2\pi}\exp\left(-\Lambda x^2/2\right)$ and $\Lambda = \lambda/2 + \sqrt{\lambda^2/4-1}$ does indeed solve (18). Unfortunately neither (17) nor (18) has a uniqueness statement. Instead, we make due with the following.

Lemma 4 Denote $\Lambda = \lambda/2 + \sqrt{\lambda^2/4-1}$. The integral equation (17) has a unique solution $f = f_*$ satisfying the side condition $\int_{-\infty}^{\infty}\exp\left(x^2/2\Lambda\right)f(x)dx < \infty$ given by $f_*(x) \equiv \sqrt{\Lambda/2\pi}\exp\left(-\Lambda x^2/2\right)$. However, it admits infinitely many solutions of less rapid decay. The analogous statement holds for (18) and the product $f_* \otimes f_*$.

Given that the "correct" solution has the best decay we can then verify the Proposition and thus the limit of the scaled one dimensional $\mathbf{M}_L$ marginal.

Proof of Lemma 3 The distribution function $\mu_{L^2}(\sigma_{L^2/2} \le x)$ is given by the integral:

$$\begin{aligned}
&\frac{1}{z_{L^2}}\int_{\{\sum_{-L^2/2}^{L^2/2}\sigma_i^2=1,\sigma_{L^2/2}\le x\}}\exp\left[DL^3\{H_{L^2-1}(\sigma)+\frac{D}{4}\sigma_{L^2/2}^4+\sigma_{L^2/2}\sigma_{L^2/2-1}\}\right]d\sigma\\
&= \frac{1}{z_{L^2}}\int_{-1}^{x}(1-z^2)^{\frac{L^2-3}{2}}\exp\left(L^3\frac{D^2}{4}z^4\right)\int_{\sum_{-L^2/2}^{L^2/2-1}\sigma^2=1}\exp\left[DL^3H_{L^2-1}(\sigma)\right]\\
&\qquad\times\exp\left[DL^3\left(-\frac{1}{2}z^2\lambda_{L^2-1}(\sigma)+z\sqrt{1-z^2}\sigma_{L^2/2}+\frac{D}{4}z^4\sum\sigma_i^4\right)\right]dzd\sigma
\end{aligned}$$

with the obvious notation H_{L^2-1} referring to the ensemble with one less particle. The density of interest then satisfies

$$\begin{aligned}
f_{L^2}(x) &= \mu_{L^2}\left(L^{3/2}\sqrt{D}\sigma_{L^2/2}=x\right)\\
&= \frac{z_{L^2-1}}{\sqrt{D}L^{3/2}z_{L^2}}\,\mathbf{E}^{\mu_{L^2-1}}\left[e^{-\lambda_{L^2-1}(\sigma)x^2/2+L^{3/2}\sqrt{D}x\sigma_{L^2/2-1}}\right](1+\mathrm{O}(1/L))
\end{aligned}$$

for $L\uparrow\infty$. This, in turn, is schematized as

$$f_{L^2}(x) \simeq f_{L^2}(0)\exp\left(-\lambda x^2/2\right)\int_{-\infty}^{\infty}\exp\left(xz\right)f_{L^2-1}(z)dz, \quad (19)$$

where the fact that (Lemma 1) $\lambda_{L^2-1}(\sigma)$ converges in law to a constant $\lambda_\infty = \lambda$ is used.

By Lemma 2, $\limsup_{L\uparrow\infty}\mathbf{E}^{\mu_{L^2}}\left[\exp\left(L^{3/2}\gamma\sigma_{L^2/2}\right)\right] < \infty$ for any γ, so, for fixed x, $\int e^{xy}f_{L^2}(y)dy$ converges along a subsequence. It then follows that $f_{L^2}(0) =$

$L^{-3/2} z_{L^2-1} / \ (\sqrt{D} z_{L^2})$ is bounded both above and away from 0 for $L \uparrow \infty$. Pointwise convergence of the density is then obtained by differentiating (19) to find

$$|f'_{L^2}(x)| \leq f_{L^2}(0)(1+\lambda|x|) \int_{-\infty}^{\infty} \exp\left(|x+\delta|z\right) f_{L^2-1}(z) dz$$

with a small $\delta > 0$. Again, for bounded x, we have control of the integral on the right, and so also $f'_{L^2}(x)$ for $L \uparrow \infty$. Arzela-Ascoli then provides a sequence over which the density functions f_{L^2} converge uniformly on compacts. Putting these comments together yield the advertised integral equation for the limiting density. The derivation for the pair density is much the same. ■

Proof of Lemma 4[7] The proof is made in the one dimensional setting (17), it is the same for (18). Integrating (17) produces $1 = f(0)\sqrt{2\pi/\lambda} \int_{-\infty}^{\infty} \exp\left(x^2/2\lambda\right) f(x) dx$, providing some control of the tails of f. A sharper control is easily obtained. Introduce the sequence

$$\Lambda_1 = \lambda, \ \Lambda_2 = \lambda - \frac{1}{\lambda}, \ldots \ \Lambda_n = \lambda - \frac{1}{\Lambda_{n-1}} \text{ where } \Lambda_\infty = \lambda/2 + \sqrt{\lambda^2/4 - 1} \equiv \Lambda.$$

Now, if one first multiplies both sides of (17) by $\exp\left(x^2/2\Lambda_n\right)$ and then integrates, it is seen that

$$\int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2\Lambda_n}\right) f(x) dx = f(0)\sqrt{2\pi/\Lambda_{n+1}} \int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2\Lambda_{n+1}}\right) f(x) dx. \tag{20}$$

That is, $\exp\left(x^2/2\Lambda_n\right) \ f(x) \in L^1$ for any $n < \infty$.

The point of the proof is in fact to show solutions of (17) split into two classes depending upon whether the monotone limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2\Lambda_n}\right) \ f(x) \ dx = \int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2\Lambda}\right) \ f(x) \ dx \tag{21}$$

is finite or not. This is achieved by iteration. Making use of the identity

$$\frac{1}{\Lambda_{n-1}} - \frac{1}{\Lambda_n} = \frac{1}{(\Lambda_1 \Lambda_2 \ldots \Lambda_{n-1})^2 \Lambda_n},$$

the n-th iterate of (17) has the form

$$f(x) \quad = \quad c_n \exp\left(-\frac{\Lambda_n}{2} x^2\right) \int_{-\infty}^{\infty} \exp\left(xz/\Lambda_1\Lambda_2\ldots\Lambda_{n-1}\right) \exp\left(z^2/2\Lambda_{n-1}\right) f(z) dz$$

with c_n depending on $f(0)$ and n. Reading the above for $x = 0$, we see that we can re-normalize things as in

$$f(x) \quad = \quad f(0) \exp\left(-\frac{\Lambda_n}{2} x^2\right) \frac{\int_{-\infty}^{\infty} \exp\left(xz/\Lambda_1\Lambda_2\ldots\Lambda_{n-1}\right) \ \exp\left(z^2/2\Lambda_{n-1}\right) f(z) dz}{\int_{-\infty}^{\infty} \exp\left(z^2/2\Lambda_{n-1}\right) f(z) dz}$$

[7] H.P. McKean helped with this.

$$\equiv \quad f(0)\exp\left(-\frac{\Lambda_n}{2}x^2\right)\int_{-\infty}^{\infty}\exp(xz)d\mu_n(z). \tag{22}$$

Now, if $\int \exp(\frac{x^2}{2\Lambda})f(x)dx < \infty$, taking limits on both sides of (20) implies $f(0) = \sqrt{\Lambda/2\pi}$. From here, (22) would provide the inequality $f(x) \geq \sqrt{\Lambda/2\pi}\exp(-\Lambda_n x^2/2)$ and so also $f(x) \geq f_*(x)$. Since both f and f_* are probability densities, the conclusion is that $f = f_*$.

If on the other hand $\int \exp(x^2/2\Lambda)dx = \infty$, we consider (22) in the limit:

$$f(x) = f(0)\exp(-\Lambda x^2/2)\int_{-\infty}^{\infty}\exp(xz)d\mu(z), \tag{23}$$

where $\mu = \mu_\infty$, the convergence of the latter being plain. From (23) we infer the rules

$$f(x) = \frac{\exp(-\Lambda x^2/2)}{\sqrt{2\pi/\Lambda}} \circ d\mu(x) = f_*(x) \circ d\mu(x) \tag{24}$$

($\circ$ = convolution) and

$$d\mu(x) = \frac{\exp(-x^2/2\Lambda)}{f(0)\sqrt{2\pi/\Lambda}}d\mu(x/\Lambda). \tag{25}$$

To verify (24), apply our integral operator to $f_* \circ d\mu$: recalling the identity $\lambda = \Lambda + 1/\Lambda$,

$$\begin{aligned} & f(0)\exp(-\lambda x^2/2)\int_{-\infty}^{\infty}\exp(xy)\ \sqrt{\Lambda/2\pi}\int_{-\infty}^{\infty}\exp(-\Lambda(y-z)^2/2)\ d\mu(z)\ dy \\ &= \quad f(0)\exp[-(\lambda - 1/\Lambda)x^2/2]\int_{-\infty}^{\infty}\exp(xz)d\mu(z) = f(x). \end{aligned}$$

For (25), we combine (23) and (24) as in

$$\begin{aligned} \int_{-\infty}^{\infty}\exp(xz)d\mu(z) &= \frac{1}{f(0)\sqrt{2\pi/\Lambda}}\ \exp(\Lambda x^2/2)\int_{-\infty}^{\infty}\exp(-\Lambda(x-z)^2/2)d\mu(z) \\ &= \int_{-\infty}^{\infty}\exp(xz)\ \frac{\exp(-z^2/2\Lambda)}{f(0)\sqrt{2\pi/\Lambda}}d\mu(z/\Lambda). \end{aligned}$$

We now generate infinitely many solutions of (17). By (24), $\mu[0] > 0$ is equivalent to $f(0) = \sqrt{\Lambda/2\pi}$ which implies $f = f_*$. Therefore, let μ have positive mass at some point $q \neq 0$. Since f is even $\mu[q] = \mu[-q]$, (25) implies both $\pm\Lambda q$ and $\pm q/\Lambda$ have positive μ mass with

$$\frac{\mu[\pm\Lambda q]}{\mu[q]} = \frac{\exp(-\Lambda q^2/2)}{f(0)\sqrt{2\pi/\Lambda}} \quad \text{and} \quad \frac{\mu[\pm q/\Lambda]}{\mu[q]} = \frac{\exp(q^2/2\Lambda)}{f^{-1}(0)\sqrt{\Lambda/2\pi}}.$$

This proliferates, μ has a sequence of atoms at $\pm\Lambda^n q$ and $\pm q/\Lambda^n$ for $n \geq 1$:

$$d\mu(x) \equiv d\mu_q(x) = \mu[q]\sum_{n=1}^{\infty}\sum_{\pm}\omega_n\delta(x \pm \Lambda^n q) + \mu[q]\sum_{n=1}^{\infty}\sum_{\pm}\bar{\omega}_n\delta(x \pm \Lambda^{-n} q), \quad (26)$$

where

$$\omega_n = \frac{\exp\left[-\frac{q^2}{2}\frac{\Lambda^{2n+1}}{\Lambda - 1/\Lambda}\right]}{f(0)\sqrt{2\pi/\Lambda}} \quad \text{and} \quad \bar{\omega}_n = \frac{\exp\left[\frac{q^2}{2}\frac{\Lambda^{-2n+1}}{\Lambda - 1/\Lambda}\right]}{f^{-1}(0)\sqrt{\Lambda/2\pi}}.$$

It is readily checked that $\mu_q[\mathbb{R}] < \infty$, and so $f_* \circ \mu_q$ for *any* $q \neq 0$ solves (17). Even more solutions arise from convex combinations of μ_q's. The proof is finished. ■

Proof of Proposition 1 The proof hinges on an integrability condition enforced by the ensemble $\mathbf{M}_L$. Recall the scaled marginal density can be written

$$\mathbf{M}_L[\sqrt{L}Q(0) = a] \simeq \frac{\mathbf{M}_L[0]}{\sqrt{L}}E^{\mu_{L^2}}\left[\exp\left\{-\lambda_{L^2}a^2/2 + L^{3/2}\sqrt{D}(\sigma_{-L^2/2} + \sigma_{L^2/2})a\right\}\right]$$

up to multiplicative errors $1 + \mathrm{O}(1/L)$, provided $|a| \leq K\sqrt{L}$ for a large K. Now for D large $L^{-1/2}\mathbf{M}_L[0]$ is bounded below, and integrating the last display in a over an appropriate range tending to the whole line as $L \uparrow \infty$ implies that any limiting density $f(x, y)$ must also satisfy

$$\int_{-\infty}^{\infty}\exp(-\lambda a^2/2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(ax + ay)f(x, y)dxdyda < \infty. \quad (27)$$

This holds when $f(x, y) = f_*(x)f_*(y)$, and next we show this is the only f for which it does.

As in Lemma 4, any potential limiting density satisfies $f(x, y) = [f_*(x)f_*(y)] \circ d\mu(x, y)$ with

$$d\mu(x, y) = \frac{\exp(-(x^2 + y^2)/2\Lambda)}{f(0, 0)2\pi/\Lambda}d\mu(x/\Lambda, y/\Lambda). \quad (28)$$

Also, $f(x, y) = f_*(x)f_*(y)$ is equivalent to $\mu = \delta_{(0,0)}$, this being the most rapid decay possible. A different solution, $f_{p,q} = f_*f_* \circ \mu_{p,q}$, may be formed by placing positive mass at a point (p, q) off the origin. Using the rule (28) to fill out the measure $\mu_{p,q}$ and the symmetries which any $f_{L^2}(x, y)$ must possess, one concludes that

$$f_{p,q}(x, y) = \frac{1}{z}\left\{\sum_{n=1}^{\infty}\omega_n\left[e^{-\frac{1}{2}\Lambda|(x,y)+\Lambda^n(p,q)|^2} + e^{-\frac{1}{2}\Lambda|(x,y)-\Lambda^n(q,p)|^2}\right]\right.$$
$$\left. + \sum_{n=1}^{\infty}\bar{\omega}_n\left[e^{-\frac{1}{2}\Lambda|(x,y)+\Lambda^{-n}(p,q)|^2} + e^{-\frac{1}{2}\Lambda|(x,y)-\Lambda^{-n}(q,p)|^2}\right]\right\} \quad (29)$$

where $\omega_n = r^n c_n$, $\bar{\omega_n} = r^{-n} c_{-n}^{-1}$ with $c_n = \exp\left(-\Lambda^{2n}(p^2+q^2)/2(\Lambda - 1/\Lambda)\right)$, $r = 2\pi f(0,0)/\Lambda < 1$, and z normalizes μ. Next let $pq > 0$ and compute: with $\Gamma_n(x,y) = (x - \Lambda^n p)^2 + (y - \Lambda^n q)^2$,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} e^{-\lambda a^2/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(ax + ay\right) f_*(x) f_*(y) \circ \mu(dx, dy) \\
&\geq \frac{1}{z}\sum_{n=1}^{\infty} \omega_n \int_{-\infty}^{\infty} e^{-\lambda\ a^2/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(ax+ay\right) \exp\Big[-\frac{\Lambda}{2}\Gamma_n(x,y)\Big] \frac{dxdy}{(2\pi/\Lambda)} da \\
&= \frac{1}{z}\sum_{n=1}^{\infty} \omega_n \int_{-\infty}^{\infty} \exp\big(-a^2/2(\Lambda - 1/\Lambda) + \Lambda^n a(p+q)\big) da \\
&= \frac{1}{z}\sum_{n=1}^{\infty} r^{-n} \exp\big(\Lambda^{2n} pq/(\Lambda - 1/\Lambda)\big) = \infty
\end{aligned}
$$

since $\Lambda > 1$. The argument with $pq \leq 0$ is similar. It only remains to note that it is sufficient to consider measures generated by a single point mass. Plainly if (27) fails for such a measure, it also fails for convex combinations. There are also f's for which μ is absolutely continuous. But these must satisfy an integrated version of (25), and our argument works then as well with slight changes. ■

4 Limiting Joint Density

We can now establish the limit of the scaled density (5). In terms of the p's this object splits into two components, recall (15). The asymptotics of the first piece are given by the following lemma, after which the proof of Theorem 1 is completed.

Lemma 5 For fixed a, b, I and $L \uparrow \infty$,

$$
\begin{aligned}
\frac{1}{L^{3/2}} p\Big(x, \frac{a}{\sqrt{L}}, \frac{b}{\sqrt{L}}, \frac{I}{L} + \frac{a^2}{L^2}\Big) \ &\simeq \ \frac{x^{3/2}}{8\pi I^2}[\phi]^{-1/2} \exp\Big[-\frac{1}{2}(a^2+b^2)(1 + \frac{x}{2I} - \phi)\Big] \\
&\quad \times L^{1/2} \exp\Big[-L\Big\{I - I\phi + \frac{x}{2}\ln\Big(\frac{x}{4I} + \frac{1}{2}\phi\Big)\Big\}\Big]
\end{aligned}
$$

with $\phi = \phi(x, I) = \sqrt{1 + \frac{x^2}{4I^2}}$.

Proof. This is very much like an old computation of Berlin and Kac [1]. Some manipulation shows that, up to exponentially small errors, $L^{-3/2}p(\cdot)$ is equal to $c_L \times P_L(a,b,I)$ with "constant"

$$
c_L = (2\pi)^{-Lx/2} I^{(Lx-3)/2} L e^{-LI} e^{-(a^2/2 + b^2/2)}
$$

and $P_L(a,b,I)$ the integral

$$\int_{\sum \sigma_i^2 = L} \exp\Big[\frac{I^2}{4L^3}\sum_{i=1}^{Lx-1}\sigma_i^4 + I\sum_{i=1}^{Lx-2}\sigma_i\sigma_{i+1} + \sqrt{I}(a\sigma_1 + b\sigma_{Lx-1})\Big]d\sigma$$

$$\simeq \int_{R^{Lx-1}}\Big\{\frac{1}{2\pi i}\int_{z_0-i\infty}^{z_0+i\infty} e^{z(L-\sum \sigma_i^2)}dz\Big\}\exp\Big[I\sum_{i=1}^{Lx-2}\sigma_i\sigma_{i+1} + \sqrt{I}(a\sigma_1 + b\sigma_{Lx-1})\Big]d\sigma$$

$$= \frac{1}{2\pi i}\int_{z_0-i\infty}^{z_0+i\infty} e^{Lz}\int_{\mathbb{R}^{Lx-1}}\exp\Big[(I\sum_{i=1}^{Lx-2}\sigma_i\sigma_{i+1} - z\sum\sigma_i^2) + \sqrt{I}(a\sigma_1 + b\sigma_{Lx-1})\Big]dxdz.$$

Here z_0 is real and chosen large enough so that the form $I\sum\sigma_i\sigma_{i+1} - z\sum\sigma_i^2$ is negative definite, justifying the exchange of integration. The integration over $\mathbb{R}^{Lx-1}$ can then be performed by diagonalizing $\sum\sigma_i\sigma_{i+1}$: it has eigenvalues $\lambda_k = \cos(\pi k/m+1)$ and eigenvectors $v_{k,j} = \sqrt{2/m}\sin(\pi jk/m+1)$ for $k,j = 1,...m = \lceil Lx-1\rceil$. Therefore, $P_L(a,b,I)$ is asymptotic to

$$(2\pi/I)^{\frac{Lx-3}{2}}\frac{1}{2\pi i}\int \exp\Big[L\Big(Iz - \frac{x}{2Lx}\sum_1^{Lx-1}\ln(z-\lambda_k)\Big) + \sum_1^{Lx-1}\frac{(av_{k,1}+bv_{k,Lx-1})^2}{4(z-\lambda_k)}\Big]dz$$

$$\simeq (2\pi/I)^{\frac{Lx-3}{2}}\frac{1}{2\pi i}\int e^{L(Iz-\frac{x}{2\pi}\ln[\frac{1}{2}(z+\sqrt{z^2-1})])}e^{\frac{a^2+b^2}{\pi}[z-\sqrt{z^2-1}]}\Big[\frac{z+1}{z-1}\Big]^{1/4}dz$$

where the contour of integration runs from $z_0 - i\infty$ to $z_0 + i\infty$. In going from the first to second line we have used the evaluations:

$$\lim_{L\to\infty}\frac{1}{Lx}\sum_1^{Lx-1}\ln(z-\lambda_k) = \frac{1}{\pi}\int_0^\pi \ln(z-\cos w)dw = \ln\Big(\frac{1}{2}(z+\sqrt{z^2-1})\Big),$$

and

$$\lim_{L\to\infty}\sum_1^{Lx-1}\frac{v_{k,1}^2}{(z-\lambda_k)} = \lim_{L\to\infty}\sum_1^{Lx-1}\frac{v_{k,Lx-1}^2}{(z-\lambda_k)} = \frac{2}{\pi}\int_0^\pi\frac{\sin^2 w}{(z-\cos w)}dw = 2\Big(z-\sqrt{z^2-1}\Big).$$

The term containing $\sum v_{k,1}v_{k,Lx-1}$ vanishes for $L\uparrow\infty$. Also present is the "error" term:

$$E_L(z) = Lx\frac{1}{2\pi}\int_0^\pi \ln(z-\cos w)dw - \frac{1}{2}\sum_0^{Lx-1}\ln(z-\cos(\frac{\pi k}{Lx})) + \frac{1}{2}\ln(z-1).$$

which has the limit $E_\infty(z) = (1/4)\ln[(z+1)/(z-1)]$. The proof is then completed by a typical stationary phase procedure: the saddle point $z_\bullet = \sqrt{1+x^2/4I^2}$ solves $0 = \Lambda'(z) = I - (x/2)\sqrt{z^2-1}$, and the fact that $\Lambda''(z) > 0$ for $z > 1$ keeps things from being tricky. ■

Proof of Theorem 1 Bring in the second piece of the density in (15), to wit,

$$\begin{aligned}
&\frac{1}{\sqrt{L}\mathbf{Z}_L} p(L-x, \frac{b}{\sqrt{L}}, \frac{a}{\sqrt{L}}, DL - \frac{I}{L} - \frac{a^2}{L}) \\
&= \quad \frac{p(L,0,0,LD)}{\sqrt{L}Z_L} \times \frac{(2\pi)^{Lx/2} p(L-x,0,0,LD)}{(DL^3)^{Lx/2} p(L,0,0,LD)} \times \\
&\quad \mathbf{E}^{\mu_{L^2-Lx}}\Big[\exp\Big\{\frac{1}{2}(LI + a^2 + b^2)\lambda_{L^2-Lx} + L^{3/2}\sqrt{D}(a\sigma_{-(L^2-Lx)/2} + b\sigma_{(L^2-Lx)/2})\Big\}\Big] \\
&\equiv \quad \Gamma_L \times \mathbf{E}^{\mu_{L^2-Lx}}[\text{etc}]
\end{aligned}$$

up to asymptotically small errors. Next combine this with the result of Lemma 5:

$$\begin{aligned}
&\mathbf{M}_L\Big[\sqrt{L}Q_0 = a,\ \sqrt{L}Q_{Lx} = b,\ \sum_1^{Lx-1} [\sqrt{L}Q_k]^2 \frac{1}{L} = I\Big] \\
&\simeq \quad \Gamma_L \frac{x^{3/2}}{4\pi I^2}(1 + \frac{x^2}{4I^2})^{-1/4} \exp\Big[-\frac{1}{2}(a^2+b^2)\Big(\Lambda + \frac{x}{2I} - \sqrt{1+\frac{x^2}{4I^2}}\Big)\Big] \\
&\quad \times\sqrt{L}\ \exp\Big[-L\Big\{\frac{\lambda}{2}I - I\sqrt{1+\frac{x^2}{4I^2}} + \frac{x}{2}\ln(\frac{x}{2I} + \sqrt{1+\frac{x^2}{4I^2}})\Big\}\Big] \\
&\quad \times\mathbf{E}^{\mu_{L^2-Lx}}\Big[\exp\Big\{\frac{1}{2}LI(\lambda - \lambda_{L^2-Lx,D}(\sigma)) + \frac{1}{2}(a^2+b^2)(\Lambda - \lambda_{L^2-Lx}(\sigma))\Big\} \\
&\qquad \exp\Big\{L^{3/2}\sqrt{D}(a\sigma_{-(L^2-Lx)/2} + b\sigma_{(L^2-Lx)/2})\Big\}\Big] \\
&\equiv \quad \Gamma_L \Phi(I,x)\Big[\frac{\rho_{x,I}}{\pi}\exp\big[-\rho_{x,I}(a^2+b^2)\big]\Big]\Big[\sqrt{L}\exp\big[-L\Psi(I)\big]\Big]E[a,b,I]. \qquad (30)
\end{aligned}$$

for fixed a, b and I and $L \uparrow \infty$. Here $\lambda = \lambda_{\infty,D}$ and you will recall $\Lambda = \lambda/2 + \sqrt{\lambda^2/4 - 1}$. In (30) we have normalized in the a, b variables in that $G_I(a)G_I(b) \equiv (\rho_{x,I}/\pi)\exp[-\rho_{x,I}(a^2+b^2)]$ is the density of two independent centered Gaussians with mean-square one over $\Lambda + \frac{x}{2I} - \sqrt{1+\frac{x^2}{4I^2}}$. Regarding the I variable, the function

$$\Psi(I) = \frac{\lambda}{2}I - I\sqrt{1+\frac{x^2}{4I^2}} + \frac{x}{2}\ln\Big(\frac{x}{2I} + \sqrt{1+\frac{x^2}{4I^2}}\Big)$$

is strictly convex with minimum at $I_0 \equiv (x/2)(\lambda^2/4 - 1)^{-1/2}$; it follows that the measure

$$\mathcal{I}_L(I)dI \equiv \sqrt{L\Psi''(I_0)/2\pi}\exp\Big[-L[\Psi(I) - \Psi(I_0)]\Big]dI$$

converges weakly to the unit mass at I_0. At $I = I_0$, $\Psi(I_0) = (x/2)\ln\Lambda$ and $\rho_{x,I_0} = \sqrt{\lambda^2/4 - 1} = \rho_D$, the limiting mean square in the statement.

Summarizing, the measure of interest has the form:

$$\begin{aligned}
\mathbf{M}_L[da, db, dI] \quad &= \quad \Lambda^{-\frac{Lx}{2}}\Gamma_L\Big[\Phi(I)\sqrt{2\pi/\Psi''(I)}\Big]E[a,b,I] \\
&\quad \times G_I(a)G_I(b)\mathcal{I}_L(I)dadbdI, \qquad (31)
\end{aligned}$$

where $G_I(a)G_I(b)\mathcal{I}_L(I)dadbdI$ converges to the advertised limiting measure. It remains to explain why the other factors fall into line for $I = I_0$ and $L \uparrow \infty$. That $E[a, b, I] \simeq 1$ for $L \uparrow \infty$ follows from $\lim_{L\uparrow\infty} \mathbf{E}^{\mu_{L^2}}[\exp L^{3/2}\sqrt{D}(a\sigma_{-L^2/2} + b\sigma_{L^2/2})] = \exp[(a^2 + b^2)/2\Lambda]$ (Proposition 1) along with the relation $\Lambda + \Lambda^{-1} = \lambda$. Next one simply checks that the function $\sqrt{\Psi''(I)/\Phi(I)}$, which is continuous and bounded for $I > 0$ equals $\sqrt{\rho_D}$ at $I = I_0$. This produces the constant factor,

$$\begin{aligned}\frac{\sqrt{\pi}}{\rho_D}\Lambda^{-\frac{Lx}{2}}\Gamma_L &= \frac{\mathbf{M}_L[0]}{\sqrt{L}}\frac{\sqrt{\pi}}{\rho_D} \times \frac{(2\pi)^{Lx/2}p_L(L-x,0,0,LD)}{(DL^3)^{Lx/2}\Lambda^{Lx/2}p_L(L,0,0,LD)} \\ &= \frac{\mathbf{M}_L[0]}{\sqrt{L}}\frac{\sqrt{\pi}}{\rho_D} \times \prod_{k=0}^{Lx-1}\frac{\sqrt{2\pi}\; z_{L^2-k-1}}{\sqrt{D\Lambda}L^{3/2}\; z_{L^2-k}}. \qquad (32)\end{aligned}$$

By Proposition 1 we know that $\mathbf{M}_L[0]/\sqrt{L} \to \sqrt{\rho_D/\pi}$. Also, looking through the proof of that Proposition will show why $\sqrt{D\Lambda/2\pi}L^{3/2}z_{L^2}z_{L^2-l}^{-1} \geq 1$ for $L \gg 1$. The shift $L^2 \to L^2 - k$ will not affect this appraisal, and it follows that the entirety of (32) is asymptotically less than one.

Finally, (31) is integrated against any non-negative compactly supported test function in a, b, and I, the support kept off of $I = 0$, to find that $\lim_{L\uparrow\infty}\mathbf{M}_L[da, db, dI]$ is dominated by the claimed limiting distribution. However, having already shown that $\mathbf{M}_L[\sqrt{L}Q_0 \in da]$ converges, the rotation invariance of the scaled ensemble implies tightness of the joint density and the inequality suffices. The proof is finished. ■

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Appendix

Proof of Lemma 1 For convenience $L^2/2$ is replaced by the integer n. The concentration statement follows from straightforward estimates; the details are not reported.

(a) Let $\sigma_k = \theta^{-1}q^{|k|}$ where $q < 1$ and $\theta = (1 + 2\sum_1^n q^{2k})^{1/2}$ keeps you on the sphere. Then,

$$H_n(q) = \frac{D}{4\theta^4}(1 + 2\sum_1^n q^{4k}) + \frac{2q}{\theta^2}\sum_0^{n-1} q^{2k} \simeq \frac{D}{4}\frac{(1-q^2)(1+q^4)}{(1+q^2)^3} + \frac{2q}{1+q^2} \equiv h(q).$$

for large n. Next one checks that $h(q)$ tends to 1 as $q \to 1$ and that $h'(1) = -\frac{D}{8}$. In fact, an expansion of $h(q)$ at $q = 1$ will show that for $D << 1$ one has the lower bound $m_{n,D} \geq 1 + D^2/32$. For $D \geq \sqrt{3}$ let $\sigma_{|k|} = D^{-|k|}$ when $|k| \geq 1$ and $\sigma_0^2 = 1 - 2\sum_1^n D^{-2k}$. A little algebra provides the desired expression.

(b) First, it is clear that at maximum all the σ_k are of one sign, so from here on we take them positive. The task is to maximize

$$H_n(\sigma) = \frac{D}{4}\sum_i \sigma_i^4 + \frac{1}{2}\sum_{i,j}\Gamma_{i,j}\sigma_i\sigma_j$$

on $\sum_{-n}^{n}\sigma_k^2 = 1$ for $\Gamma_{i,j} = 1$ if $|i-j| \le 1$ and 0 otherwise. The point of this disguise is as follows. Both the sum of fourth powers and the constraint, $\sum \sigma_i^2 = 1$, are insensitive to permutations of the indices. The term $\sum \Gamma_{i,j}\sigma_i\sigma_j$ is a different matter. Since $\Gamma_{i,j} = \Gamma(|i-j|)$ is 'radial' and decreasing, a rearrangement theorem of Riesz[8] says that to make this last term largest the maximizer should peak at the center ($n = 0$) and be increasing/decreasing to the left/right of $n = 0$.

By the shape, $\sigma_n \le \sigma_{n-1}\cdots$ etc., one knows at least $\sigma_{|k|} \le |k+1|^{-1/2}$. This, along with the Euler-Lagrange equations,

$$\sigma_{k-1} + \sigma_{k+1} = \lambda\sigma_k - D\sigma_k^3 \quad \text{for} \quad -n < k < n$$

($\sigma_{n+1} = \sigma_{-n-1} = 0$) gives $\sigma_{n-1} \ge (\lambda - D/(n+1))\sigma_n$. Upon iteration this is

$$\sigma_{n-k(n)} \ge (\lambda - \frac{D}{n+1})\prod_{1}^{k(n)-1}(\lambda - 1 - \frac{D}{n+1-j})\sigma_n,$$

and since, for fixed D and large enough dimension, $\lambda > 2+\delta$ for $\delta > 0$ you may choose an M such for all $m \ge M$, $D/m < \delta/2$ and playing the above game up to $k = n - M$ gives the bound $\sigma_n \le (1+\delta/2)^{M-n}$. So, for $2M \le |k| \le n$ say, there are (positive) constants C and θ such $\sigma_{|k|} \le Ce^{-\theta|k|}$. Note that θ increases with D due the improved lower bounds on λ for large D from part **(a)**.

The maximizers σ^n of $H_n(\sigma)$ provide a maximizing sequence for $H_\infty(\sigma)$. As each σ^n sits on the $2n$-sphere, there exists a subsequence tending to a limit σ^∞ coordinate-wise. The exponential decay of the tails of σ^n provides the necessary domination: $H_\infty(\sigma^\infty) = m_\infty$.

(c) As to uniqueness, we have been able to handle large D because here we have reasonable lower estimates for H and so also for λ. In detail, the lower estimate for large D along with the fact that $\sigma_0 \ge \sigma_k$ at maximum yields that there $(D/4) + (3/2)D \le H_{n,D}(\sigma) \le (D/4)\sigma_0^2 + 1$. This, in turn, implies $\sum_{k\neq 0}\sigma_k^2 = 1 - \sigma_0^2 \le 4/D$, so that $\sigma_{\pm 1} \le 4/D$ and in fact $\sigma_{\pm|k|} \le const. \times D^{-|k|}$. The important thing to take away is that, if there are several maximizers, they all satisfy $\sigma_0 = 1 - \mathrm{O}(1/D)$.

Now suppose there are two maximizers (σ^0 and σ^2) and argue by contradiction. Think of H as a function of the angle $\theta : 0 \le \theta \le \theta_1$ along the geodesic between these points: $\sigma^0 = \sigma(0)$ and $\sigma^1 = \sigma(\theta_1)$ with $h(\theta) = H(\sigma(\theta))$. Next, look at the action of a small rotation along this arc upon h. Noting that $\sigma''(\theta) = -\sigma(\theta)$ and

[8] See [4]. We thank Professor Elliot Lieb for pointing this out to us.

$||\sigma'(\theta)|| = 1$, we compute:

$$h''(\theta) = -\lambda + 3D\sum \sigma_k^2(\sigma_k')^2 + 2\sum \sigma_k'\sigma_{k+1}' \le -\lambda(\theta) + 3D(\sigma_0')^2(\theta) + \mathrm{O}(1)$$

where the inequality follows from Cauchy-Schwartz along with the bound $\sigma_{\pm|k|} \le 4/D$ for $|k| \ge 1$. This rotation certainly keeps us in a neighborhood of $\sigma_0 = 1$ (equivalently $\sum_{k\neq 0}\sigma_k^2 \simeq 4/D$), and this implies that $\lambda(\theta) \ge D\sigma_0^4(\theta) \simeq D(1 - \mathrm{O}(1/D))$ and $(\sigma_0')^2(\theta) \simeq \sin^2(\theta) \simeq \mathrm{O}(1/D)$ along the arc. Using these estimates in (A.3) shows that $h''(\theta) < 0$ for all large enough D, but that contradicts the assumption that $h(0)$ and $h(\theta_1)$ are maximal.

The convergence of λ_{L^2} to a constant in probability for large D follows from uniqueness of the maximizer along with the observation that $\lambda_\infty(\sigma)$ is invariant under translations ($\sigma_k \to \sigma_{k+m}$) and reflections ($\sigma \to -\sigma$). The proof is finished. ■

Proof of Lemma 2 Wanting to control $\mathbf{E}^{\mu_L^2}[\exp L^3\gamma\sigma_{L^2}^2]$ we may assume from (12) that $\sigma_{L^2} < L^{-5/6}$ say. Next, repeating the calculation starting the proof of Lemma 3: with some constant C and $L \uparrow \infty$,

$$\begin{aligned}\mathbf{E}^{\mu_L^2}[e^{L^3\gamma\sigma_{L^2}^2}] \quad &\le \quad \frac{C}{z_{L^2}}\int_{S_1^{L^2-2}} d\sigma \exp\left[L^3 D H_{L^2-1}(\sigma)\right] \\ &\quad \times \int_{-\infty}^{\infty} d\sigma_{L^2} \exp\left[L^3\gamma\sigma_{L^2}^2 - L^3 D\frac{1}{2}\lambda_{L^2-1}(\sigma)\sigma_{L^2}^2 + L^3 D\sigma_{L^2}^2\right]\end{aligned}$$

where we have also restricted to the set $\sigma_{L^2-1} \le \sigma_{L^2}$. The Gaussian integration in σ_{L^2} may then be performed if γ is chosen so that $D\lambda/2 - D - \gamma > 0$ for $L \uparrow 0$ which may be done the results of Lemma 1 (**a**). The whole right side is then bounded by use of the simple estimate seen before: $z_{L^2} \ge const.L^{-3/2}z_{L^2}$. For the integral over the set $\sigma_{L^2} < \sigma_{L^2-1}$ one simply iterates the same argument to complete the proof.

As for $L^{-1/2}\mathbf{M}_L[0]$, we first mention that one has an upper bound for all D. Examining the inverse, a little manipulation will show $\sqrt{L}\mathbf{M}_L[0]^{-1} =$

$$\begin{aligned}\frac{\sqrt{L}\mathbf{Z}_L}{p(L,0,0,LD)} \quad &\ge \quad D^{1/2}L^{3/2}\int_{-\delta}^{\delta}(1-c^2)^{(L^2-3)/2}\mathbf{E}^{\mu_L^2}\left[\exp\left[-L^3 D\lambda_{L^2}(\sigma)c^2/2\right]\right] \\ &\quad \times \exp\left[DL^3(c(1-c^2)^{1/2}(\sigma_{-L^2/2} + \sigma_{L^2/2}))\right]\Big] dc,\end{aligned}$$

for any $\delta > 0$ but less than 1. Next, Jensen's inequality is applied in the μ_{L^2} expectation to produce: with $\mathbf{E}^{\mu_{L^2}}[\sigma_k] = 0$ for any k,

$$\begin{aligned}\sqrt{L}(\mathbf{M}_L[0])^{-1} \quad &\ge \quad D^{1/2}L^{3/2}(1-\delta^2)^{L^2/2}\int_{-\delta}^{\delta}\exp\left[-DL^3c^2\mathbf{E}^{\mu_{L^2}}[\lambda]/2\right]dc \\ &\ge \quad \sqrt{\frac{2\pi}{\mathbf{E}^{\mu_{L^2}}[\lambda_{L^2}]}}(1-\delta^2)^{L^2/2}\left\{1 - \frac{1}{\delta}\frac{\exp\left(-DL^3\mathbf{E}^{\mu_{L^2}}[\lambda_{L^2}]\delta^2\right)}{L^{3/2}D^{1/2}\mathbf{E}^{\mu_L^2}[\lambda_{L^2}]}\right\}.\end{aligned}$$

Here we have used the fact $\mathbf{E}^{\mu_{L^2}}[\lambda_{L^2}(\sigma)] \geq 2\mathbf{E}^{\mu_{L^2}}[H_{L^2}(\sigma)] > 0$ for L large. Taking $\delta = \delta(L) = L^{-3/4}$ and $L \uparrow \infty$ will explain why a priori $\limsup_{L\uparrow\infty} \mathbf{M}_L[0]/\sqrt{L} \leq (1 + D/2)/\sqrt{2\pi}$.

An estimate of $L^{-1/2}\mathbf{M}_L[0]$ from below requires more. A necessary condition stems from integrating the density $\mathbf{M}_L[Q_0 = a]$ (recall (13)) over some large range $|a| \leq K$. By tightness we find that

$$1 - \mathrm{o}(1) \quad \leq \quad \frac{\mathbf{M}_L[0]}{\sqrt{L}} \mathbf{E}^{\mu_{L^2}} \left[\sqrt{\frac{2\pi}{\lambda_{L^2}}} \exp \left[L^3 D(\sigma_{-L^2/2} + \sigma_{L^2/2})^2 / 2\lambda_{L^2} \right] \right],$$

up to small errors on the right hand side for L large. Thus, $L^{-1/2}\mathbf{M}_L[0] = O(1)$ for $L \uparrow \infty$ is implied by

$$\limsup_{L\uparrow\infty} \mathbf{E}^{\mu_{L^2}} \left[e^{L^3 D(\sigma_{-L^2/2} + \sigma_{L^2/2})^2 / 2\lambda_{L^2}} \right] \quad \leq \quad \limsup_{L\uparrow\infty} \mathbf{E}^{\mu_{L^2}} \left[e^{2L^3 D \sigma_{L^2/2}^2 / \lambda_{L^2}} \right] < \infty;$$

allowing the desired conclusion for large D by taking $\gamma = 2$ above since in that case we know $\lambda_{L^2,D} > D$ for $L \uparrow \infty$. ■

References

[1] T.H. Berlin, M. Kac, The spherical model of a ferromagnet, *Phys. Rev.*, **86**, 1952, pp. 821-835.

[2] J. Bourgain Periodic nonlinear Schrödinger equation and invariant measures, *Comm. Math. Phys*, **166**, 1994, pp. 1-26.

[3] Y. Burlakov, The phase space of the cubic Schrödinger equation: a numerical study, *Berkeley Preprint 740*, 1998.

[4] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, *Cambridge University Press*, 1934

[5] J.L. Lebowitz, H. Rose, E. Speer, Statistical mechanics of nonlinear Scrödinger equation, *Journal of Stat. Phys.* , **50**, 1988, pp. 657-687.

[6] H.P. McKean, Statistical mechanics of nonlinear wave equations (IV): cubic Schrödinger, *Comm. Math. Phys.*, **168**, 1995, pp. 479-491.

[7] H.P. McKean, A Martin boundary connected with the ∞-volume limit of focussing cubic Schrödinger, Itô's Stochastic Calculus and Probability Theory, 251–259, Springer, 1996.

[8] H.P. McKean, K. Vaninsky, Statistical mechanics of nonlinear wave equations (1). Papers dedicated to F. John, ed. L. Sirovich. Trends and Perspectives in Applied Math., Springer, 1994.

[9] H.P. McKean, K. Vaninsky, Brownian Motion with Restoring Drift: The petit and micro-canonical ensembles *Comm. Math. Phys.* **160**, 1994, pp. 615-630.

[10] B. Rider, On the ∞-Volume limit of focussing cubic Schrödinger *Comm. Pure and Appl. Math.* **LV** 2002, 1231-1248.