A Limit Theorem at the edge of a non-Hermitian Random Matrix Ensemble

B. Rider[1]

Abstract The study of the edge behavior in the classical ensembles of Gaussian Hermitian matrices has led to the celebrated distributions of Tracy-Widom. Here we take up a similar line of inquiry in the non-Hermitian setting. We focus on the family of $N \times N$ random matrices with all entries independent and distributed as complex Gaussian of mean zero and variance $\frac{1}{N}$. This is a fundamental non-Hermitian ensemble for which the eigenvalue density is known. Using this density our main result is a limit law for the (scaled) spectral radius as $N \uparrow \infty$. As a corollary we get the analogous statement for the case where the complex Gaussians are replaced by quaternion Gaussians.

1 Introduction

Certainly some of the most striking results concerning the spectra of random matrices are those around the limiting behavior of the edge. The driving force behind this effort has of course been Tracy-Widom who obtained the convergence in distribution of the scaled largest eigenvalue in the Gaussian Orthogonal, Unitary and Symplectic Ensembles ([13], [14]) and showed that the limit laws have exact expression in terms of Painlevé II. We mention as well the work of Soshnikov [12] who has shown a type of universality at the edge for Wigner matrices with entries respecting Gaussian conditions on the growth of their moments. More recently, Johnstone [10] has repeated the Tracy-Widom analysis for the Laguerre ensemble (or Gaussian covariance matrices).[2]

However, this style of question seems to have been largely ignored for the case of non-Hermitian random matrices. There are a class of ensembles with complex spectra for which the limiting density of states is understood to be supported on some set E_∞ in the complex plane. Understanding the edge for such an ensemble entails identifying $\varepsilon_N \downarrow 0$ for $N \uparrow \infty$ for which

$$\lim_{N\uparrow\infty} \text{Prob}\Big[\text{all eigenvalues lie in } E_\infty \times (1+\varepsilon_N)\Big]$$

exists and is non-trivial. That is, one wishes to measure the distributional distance of the spectral point furthest from its limiting support as $N \uparrow \infty$. The purpose of this short note is to hopefully draw some attention to these problems and to

[1]Department of Mathematics, Duke University, Durham, NC 27708, e-mail: rider@math.duke.edu

[2]It is not our intent to slight the huge body of work connecting the largest eigenvalue of GUE etc and problems in combinatorics and the statistical mechanics of growth models, but rather to keep the introduction brief.

demonstrate that, in the particular case taken up here, elementary probability arguments lead to rather precise information concerning the spectral edge.

We will mainly consider the spectrum of perhaps the most basic non-self-adjoint random matrix ensemble. In particular, let $M^N = [M_{ij}^N]$ be the $N \times N$ random matrix whose entries are independent complex centered Gaussians of variance N^{-1}. The exact eigenvalue density was derived by Ginibre [8]:

$$\mathcal{P}_N(z_1, z_2, \dots z_N) = \frac{1}{Z_N} e^{-N\sum_{k=1}^N |z_k|^2} \prod_{1 \le j < k \le N} |z_j - z_k|^2 \tag{1}$$

for $z_k = x_k + \sqrt{-1} y_k$ and Z_N the appropriate normalizer. The well known Circular Law (see [1] or the computation of J. Silverstein appearing in [9]) states that $\frac{1}{N}\sum_{k=1}^N \delta_{z_k}$ converges weakly to the uniform measure on the unit disk $D = \{|z| \le 1\}$. Therefore, identifying an N-dependent disk which scales to D and captures all the eigenvalues is equivalent to studying the spectral radius. Our main result is the following.

Theorem 1 *Let R_N be the spectral radius for the matrix M_N. That is, $R_N = \max_{1 \le k \le N} |z_k|$. Then,*

$$\lim_{N\to\infty} \mathcal{P}_N \Big[\sqrt{4N\gamma_N} \Big(R_N - 1 - \sqrt{\frac{\gamma_N}{4N}} \Big) \le x \Big] = \exp\Big[-\exp(-x) \Big] \tag{2}$$

in which $\gamma_N = \log \frac{N}{2\pi} - 2\log\log N$. In other words, R_N is well approximated by

$$R_N \simeq 1 + \sqrt{\frac{\gamma_N}{4N}} - \frac{1}{\sqrt{4N\gamma_N}} \log Z$$

with Z an exponential random variable of mean one.

Perhaps the first thing the reader will note in the above statement is that there is none of the integrable phenomena present which has caused so much excitement around the work of Tracy-Widom. That is, the appropriate scaling is somewhat complicated while the limiting law coming out on the right is fairly simple; there is no Painlevé transcendent floating about. Second, a look at (2) indicates that the largest eigenvalue tends (with large probability) to lie outside the unit disk as $N \uparrow \infty$. It is interesting to contrast this with the known cases for real spectra: the limit distributions of the maximal eigenvalue in $G(O/U/S)E$ (after subtracting off the edge and scaling by an increasing factor) all have negative mean.

Next we wish to point out that for the present ensemble we may also obtain almost sure information about the spectral edge. The result shows that in fact $1 \le R_N \downarrow 1$ with probability one as $N \uparrow \infty$ and more.

Theorem 2 *We have the following almost sure statement for the behavior of R_N. For any $\delta > 0$ and $M < \infty$,*

$$\mathrm{Prob}\Big[1 + \frac{M}{\sqrt{N}} \le R_N \le 1 + (2+\delta)\sqrt{\frac{\log N}{N}}, N \uparrow \infty\Big] = 1. \tag{3}$$

The almost sure convergence of the absolute value of the largest eigenvalue to one does not follow from the weak convergence of the empirical spectral distribution embodied in the Circular Law. On the other hand, the Circular Law coupled with the techniques of [7] (which deals with general real entried non-Hermitian matrices) will furnish a proof of $\mathrm{Prob}[\lim_{N\uparrow\infty} R_N = 1] = 1$. The point is, (3) contains more detailed information for the present Gaussian ensemble. For some impressive results on the almost sure convergence of extremal eigenvalues to their predicted values in the Hermitian case we refer the reader to [3] (general Wigner matrices) and [2] (general Sample-Covariance matrices).

Theorems 1 and 2 should be compared to the classical result of Mehta ([11], 15.1.35) who studied the behavior of the one-point function at the edge. In particular, with $p_N(r) = \mathrm{Prob}[|z_k| = r$ for some $k]$, he has shown: with $s > 0$ and of order one,

$$p_N(1 + \frac{s}{\sqrt{N}}) \simeq \frac{1}{\pi} - p_N(1 - \frac{s}{\sqrt{N}}) \simeq \frac{e^{-s^2}}{s\sqrt{\pi}} \tag{4}$$

to leading order as $N \uparrow \infty$. This gives information on the sharpness of the edge from the point of view of the average eigenvalue and identifies the characteristic distance as roughly order $1/\sqrt{N}$. The present results strengthen this by providing the probability that *any* eigenvalue is near the edge. Note that (4) can also be found in [6] along with the corresponding two-point function in the vicinity of the unit circle.

For completeness we mention that our analysis carries over exactly to the ensembles M_Q^N in which the entries are taken to be independent quaternion Gaussians.

Corollary 1 *Let R_N^Q denote the spectral radius of the non-Hermitian matrix with independent quaternion Gaussian entries. The statements of Theorem 1 and Theorem 2 remain valid with R_N^Q in place of R_N so long as* $\exp[-\sqrt{2}\exp(-x)]$ *replaces the right hand side of (2).*

As has been pointed out to us, the result of Corollary 1 is not so plain. In the quaternion case the eigenvalues tend to avoid the real axis; one loses the rotation invariance of the complex ensemble. Despite this difference, the eigenvalue of largest modulus, regardless of whether it chances to be near or far from the real axis, scales and is distributed identically (up to a trivial constant factor) as in the complex case. It would be nice to know if the real case follows suit, but that density is harder to deal with (see [5]).

In the next section we describe a different non-Hermitian ensemble for which there is reason to believe the edge behavior will have more in common with the well known GUE type results. After this short expository effort, the proofs of Theorems 1 and 2 along with their corollary are provided in Section 3.

2 Other Ensembles with Complex Spectra

Here we simply remind the reader of another class of non-Hermitian ensemble for which we feel a study of the edge behavior would be interesting - though we have been unable to do so yet. Consider the random matrix

$$M_\tau^N = A^N + \sqrt{-1}\sqrt{\frac{1-\tau}{1+\tau}} B^N$$

in which A^N and B^N are independent copies of an $N \times N$ GUE and the parameter τ residing in $[0,1]$ allows M_τ^N to interpolate between GUE ($\tau = 1$) and the ensemble central to our work ($\tau = 0$). Again the eigenvalue density is known:

$$\mathcal{P}_N^\tau(z_1, z_2, \ldots, z_N) = \frac{1}{Z_N^\tau} \exp\Big[-\frac{N}{1-\tau^2}\sum_{k=1}^N (|z_k|^2 - \tau \mathcal{R}e z_k^2)\Big] \prod_{1 \le j < k \le N} |z_j - z_k|^2,$$

and the limiting density of states is the uniform measure on the ellipse with semi-major/minor axes $1+\tau$ and $1-\tau$.

It turns out that, for $\tau > 0$, the family of Hermite polynomials $\{H_k(z)\}$, appropriately scaled and normalized, are orthonormal with respect to the weight $w_{N,\tau}(z) = \exp[-\frac{N}{1-\tau^2}(|z|^2 - \tau\mathcal{R}e z^2)]$ in the complex plane.[3] This allows one to write the probability that all the eigenvalues lie within a certain set E as the Fredholm determinant of a kernel operator acting on the exterior of that set. Experts will recognize this is the all-important opening move in the work of Tracy-Widom. The presence of this familiar structure is encouraging. However, difficulties soon arise from two fronts. First, with the needed scaling the Hermite polynomials no longer satisfy the Christoffel-Darboux formula which clouds the rigorous analysis of the kernel. Second, there is the basic non-local nature of the problem; strong asymptotics are needed near the whole boundary of the limit ellipse as opposed to a single boundary point as in the Hermitian case. Still, we hope to return to this matter in the future.

3 Proofs

Proof of Theorem 1 The basic formula that is the starting point of our analysis is, of course, found in Mehta's book [11]. Computing the probability that R_N is

[3]See [15] for this fact and an interesting treatment of this ensemble in a different vein.

less than say a entails integrating (1) over $\{|z_k| \le a : 1 \le k \le N\}$. Noting that the integrand is symmetric in all z_k and that $\prod |z_j - z_k|^2 = \triangle(z)\triangle(\bar{z})$ with $\triangle$ the usually Vandermonde determinant, one may apply row/column reductions inside the integral to find

$$\begin{aligned}
\mathcal{P}_N(a) &= \mathcal{P}_N\Big[\max_{1\le k\le N} |z_k| \le a\Big] \\
&= \int_{|z_1|\le a} \cdots \int_{|z_N|\le a} e^{-N\sum_{k=1}^N |z_k|^2} \det[\bar{z}_i^{i-1} z_i^{j-1}]_{1\le i,j\le N} dx_1 dy_1 \cdots dx_N dy_N \\
&= \det\Big[\int_{|z|\le a} e^{-N|z|^2} \bar{z}^i z^j \, dxdy\Big]_{0\le i,j\le N-1} \\
&= \prod_{k=0}^{N-1} \frac{N^{k+1}}{\Gamma(k)} \int_0^{a^2} e^{-Nr} r^k dr = \prod_{k=0}^{N-1} \mathbf{P}\Big[\frac{1}{N}\sum_{\ell=1}^{N-k} X_\ell \le a^2\Big] \qquad (5)
\end{aligned}$$

for $\{X_k\}$ a sequence of independent exponential random variables of parameter one and $\mathbf{P}$ their corresponding measure. Note that we are able to perform the integration in line 2 since each row in the determinant depends on a distinct variable z_k. Also, $\int_0^a e^{-N|z|^2} \bar{z}^i z^j dxdy = 0$ if $i \ne j$, leaving just a diagonal determinant to evaluate in the last line.

From the Law of Large Numbers one sees immediately why $\mathcal{P}_N(a) \to 0$ or 1 if a is either strictly smaller or strictly greater than 1. To identify the appropriate scaling about $a = 1$ we write $a = 1 + \frac{1}{2\sqrt{N}} f_N(x)$ with f_N an increasing function in both x and N. With this, the factors in the product (5) are more easily analyzed if written in the form

$$\begin{aligned}
\mathbf{P}\Big[\frac{1}{N}\sum_{\ell=1}^{N-k} X_\ell \le a^2\Big] &= \mathbf{P}\Big[\frac{1}{\sqrt{N}}\sum_{\ell=1}^{N-k} (X_\ell - 1) \le f_N(x) + \frac{1}{4\sqrt{N}} f_N^2(x) + \frac{k}{\sqrt{N}}\Big] \\
&= \mathbf{P}\Big[\frac{1}{\sqrt{N}}\sum_{\ell=1}^{N-k} (X_\ell - 1) \le \phi_N(x) + \frac{k}{\sqrt{N}}\Big] \equiv p_k. \qquad (6)
\end{aligned}$$

The exact form of f_N (or $\phi_N(x)$) is just what is to be determined. Certainly $f_N = o(\sqrt{N})$, but we will assume from the start that $f_N(x) \uparrow \infty$ with $N \uparrow \infty$ for x fixed. Indeed, now the Central Limit Theorem explains why $\mathcal{P}_N(1 + \frac{1}{\sqrt{N}} M) \to 0$ for any constant M as $N \uparrow \infty$; a comment we will return to in the proof of Theorem 2.

Next, again by the CLT and a glance at (6) one expects the main contribution to stem from that part of the product with k no more than order $\sqrt{N}$. Certainly the probability in question is bounded by any truncated product, that is, for whatever positive δ_N less than one,

$$\mathcal{P}_N\Big(1 + \frac{1}{2\sqrt{N}} f_N(x)\Big) \le \prod_{k=0}^{N\delta_N} p_k.$$

What is more, for x bounded from below, the opposite inequality holds up to multiplicative errors of order $1 - O(1/N)$ if we take $\delta_N = \sqrt{2N^{-1}\log N}$. The conclusion is drawn form the following string of inequalities: for $0 < \alpha < 1$,

$$\begin{aligned}
&\prod_{k=N\delta_N}^{N-1} \mathbf{P}\Big(\frac{1}{\sqrt{N}}\sum_{\ell=1}^{N-k}(X_\ell - 1) \le \phi_N(x) + \frac{k}{\sqrt{N}}\Big) \\
&\ge \prod_{k=N\delta_N}^{N-1} \Big(1 - \mathbf{P}\Big[\sum_{\ell=1}^{N-K} X_\ell \ge \sqrt{N}\phi_N(x) + N\Big]\Big) \\
&\ge \prod_{k=N\delta_N}^{N-1} \Big(1 - e^{-\alpha\sqrt{N}\phi_N(x)}\mathbf{E}[e^{\alpha X_1}]^{N-k}\Big) \\
&\ge \prod_{k=N\delta_N}^{N-1} \Big(1 - \exp\Big[-N\Big\{\alpha(1+\frac{k}{N}) + \log(1-\alpha)\Big\} - \alpha\sqrt{N}\phi_N(x)\Big]\Big) \\
&\ge \prod_{k=N\delta_N}^{N-1} \Big(1 - \exp\Big[-N\Big\{\frac{k}{N} - \log(1+\frac{k}{N})\Big\}\Big]\Big) \ge \Big(1 - e^{-N\delta_N^2}\Big)^N.
\end{aligned}$$

The last line stems from the positivity of $\phi_N(x)$ and setting $\alpha = 1 - (1 + k/N)^{-1}$ in order to maximize the remaining exponent.

Therefore, we have that

$$\begin{aligned}
&\log \mathcal{P}_N\Big(1 + \frac{1}{2\sqrt{N}} f_N(x)\Big) \\
&= \sum_{k=1}^{\sqrt{2N\log N}} \log \mathbf{P}\Big[\frac{1}{\sqrt{N-k}}\sum_{\ell=1}^{N-k}(X_\ell - 1) \le \sqrt{\frac{N}{N-k}}(\phi_N(x) + \frac{k}{\sqrt{N}})\Big]. \qquad (7)
\end{aligned}$$

uniformly in N and x for $x > -\infty$ and $N \uparrow \infty$. The individual summands may be then analyzed by use of the classical Edgeworth expansion; the latter providing uniform corrections to the Central Limit Theorem. Let $p_M(t)$ denote the density of the random variable $\frac{1}{\sqrt{M}}\sum_{\ell=1}^{M}(X_\ell - 1)$ at t. The needed statement (see for example [4] Corollary 19.4) is that

$$\sup_{-\infty<t<\infty}\Big|p_M(t) - \frac{e^{-t^2/2}}{\sqrt{2\pi}} - \rho_1(t)\frac{e^{-t^2/2}}{\sqrt{M}\sqrt{2\pi}} - \rho_2(t)\frac{e^{-t^2/2}}{M\sqrt{2\pi}}\Big| = \mathrm{O}(M^{-3/2}) \qquad (8)$$

where $\rho_1(t) = c_1 t^3$ and $\rho_2(t) = c_2 t^4 + c_3 t^6$ with constants c_1, c_2, c_3 independent of M and expressed in terms of the moments of X_1.

With the help of (8) we find that, for x restricted as in $|x| \le L$ for some large positive L and any $K_N \uparrow \infty$ faster than $\sup_{-L\le x\le L}\phi_N(x)$

$$\log \mathbf{P}\Big[\frac{1}{\sqrt{N-k}}\sum_{\ell=1}^{N-k}(X_\ell - 1) \le \phi_N(x) + \frac{k}{\sqrt{N}}\Big] = \log\Big(\int_{-K_N}^{\phi_N(x)+\frac{k}{\sqrt{N}}} \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt\Big)$$

$$+\mathrm{O}(\frac{1}{\sqrt{N}}\sup_{|c|\leq L}\phi_N^2(c)e^{-\phi_N^2(c)/2})+\mathrm{O}(\frac{1}{N})+\mathrm{O}(\frac{K_N}{N^{3/2}})+\mathrm{O}(e^{-\frac{1}{2}\sqrt{N}K_N}). \quad (9)$$

Here we have made use of the fact that $k=o(N)$. Instead of compacting the error terms, we write the above in the present form so that the reader may check their origin. The last stems from restricting the probability on the left to the set where $\frac{1}{\sqrt{N-k}}\sum_{\ell=1}^{N-K}(X_\ell-1)\geq -K_N$. With this precaution the bound (8) is integrated over $-K_N\leq t\leq \phi_N+k/\sqrt{N}$ producing the first three error terms. Afterwards, one may extend the lower limit of integration in the leading term on the right of (9) from $-K_N$ down to $-\infty$.

Substituting the estimate (9) into (7) it is plain that we should interpret the leading order sum as the Riemann integral it approximates. That is, we should take

$$\sum_{k=1}^{\sqrt{2N\log N}}\log\Big[\int_{-\infty}^{\phi_N(x)+\frac{k}{\sqrt{N}}}\frac{e^{-t^2/2}}{\sqrt{2\pi}}dt\Big]\simeq\sqrt{N}\int_{\phi_N(x)}^{\phi_N(x)+\sqrt{2\log N}}\log\Big[\int_{-\infty}^{t}\frac{e^{-s^2/2}}{\sqrt{2\pi}}ds\Big]dt \quad (10)$$

which may be done subject to an additional error bounded above by

$$\sqrt{N}\sum_{k=1}^{\sqrt{2N\log N}}\int_{\phi_N+\frac{k}{\sqrt{N}}}^{\phi_N+\frac{k+1}{\sqrt{N}}}\log\Big(1+\int_{\phi_N+\frac{k}{\sqrt{N}}}^{s}e^{-t^2/2}dt\Big)ds\leq C\sqrt{\log N}e^{-\phi_N^2/2}.$$

Now one checks that the upper limit of integration in (10) may be extended to $+\infty$ from which one sees that $\phi_N=o(\sqrt{\log N})$ and that K_N may be chosen to be of the order of $\log N$. The conclusion is that: with x bounded in the same manner as before:

$$\log\mathcal{P}_N\Big(1+\frac{1}{2\sqrt{N}}f_N(x)\Big) \;=\; \sqrt{N}\int_{\phi_N(x)}^{\infty}\log\Big[\int_{-\infty}^{t}\frac{e^{-s^2/2}}{\sqrt{2\pi}}ds\Big]dt+E_N \quad (11)$$

where

$$E_N=O\Big((\frac{\log N}{\sqrt{N}})\vee(\sqrt{\log N}\sup_{|c|\leq L}\phi_N^2(c)e^{-\phi_N^2(c)/2})\Big). \quad (12)$$

Finally, the scaling function $f_N(x)$ and limit law, the distribution function of which we denote by $F_\infty(x)$, are identified by requiring that

$$\lim_{N\uparrow\infty}\sqrt{N}\int_{\phi_N(x)}^{\infty}\log\Big[1-\int_t^\infty e^{-s^2/2}\frac{ds}{\sqrt{2\pi}}\Big]dt=\log F_\infty(x) \quad (13)$$

point-wise in x. Note that as the right hand side is to represent the logarithm of a single distribution function, $f_N(x)$ must be chosen so that whatever comes out of the limit must increase from $-\infty$ to 0 as x ranges between $\mp\infty$. The function $f_N(x)$ is of course further restricted by the error term (12) being $o(1)$ for $N\uparrow\infty$.

Proceeding, you will note that with $0 < f_N \uparrow \infty$ the condition (13) is the same as

$$\begin{aligned}
\log F_\infty(x) &= -\lim_{N\uparrow\infty} \sqrt{\frac{N}{2\pi}} \int_{\phi_N(x)}^\infty \int_t^\infty e^{-s^2/2} ds dt \times \Big(1 + O(\int_{\phi_N(x)}^\infty e^{-s^2/2} \frac{ds}{\sqrt{2\pi}})\Big) \\
&= -\lim_{N\uparrow\infty} \sqrt{\frac{N}{2\pi}} \int_{\phi_N(x)}^\infty \frac{1}{t} e^{-t^2/2} dt \times \Big(1 + O(\frac{1}{\phi_N(x)})\Big) \\
&= -\lim_{N\uparrow\infty} \sqrt{\frac{N}{2\pi}} \frac{1}{f_N^2(x)} \exp\Big[-\frac{1}{2} f_N^2(x)\Big] \times (1 + o(1)), \qquad (14)
\end{aligned}$$

as $\phi_N = f_N \times (1 + \frac{f_N}{\sqrt{4N}})$. Now choosing

$$f_N^2(x) = 2\log\Big(\frac{e^x\sqrt{N/2\pi}}{\log N}\Big)$$

we find that uniformly on compact sets in x:

$$\begin{aligned}
&\lim_{N\uparrow\infty} \log \mathcal{P}_N \left[R_N \le 1 + \sqrt{\frac{1}{2N}} \Big(\log \frac{\sqrt{N/2\pi}}{\log N} + x\Big)^{1/2} \right] \\
&= -\lim_{N\uparrow\infty} \Big(\sqrt{\frac{N}{2\pi}} \frac{1}{f_N^2(x)} \exp\Big[-\frac{1}{2} f_N^2(x)\Big] \times (1 + O(\frac{1}{\sqrt{\log N}})) + O(\frac{(\log N)^3}{\sqrt{N}})\Big) \\
&= -\exp[-x]. \qquad (15)
\end{aligned}$$

That (15) entails an equivalent limit theorem to that in the statement (2) follows from standard properties of distribution functions. ■

Proof of Theorem 2 This is a reprise of the type of estimates employed above. For the left hand part of the statement in (3), recall (6) and note the inequality

$$\mathcal{P}_N \Big[R_N \le 1 + \frac{M}{\sqrt{N}} \Big] \le \prod_{k=1}^{\sqrt{N}} \mathbf{P}\Big[\frac{1}{\sqrt{N}} \sum_{\ell=1}^{N-k} (X_\ell - 1) \le 2M \Big].$$

Now each term in the product converges to $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2M} \exp -c^2/2 dc < 1$, and by the Berry-Essen Theorem ([4] Theorem 12.4) the rate of convergence is bounded by a constant multiple of $N^{-1/2}$. Therefore, there exists some positive δ for which

$$\mathcal{P}_N \Big[R_N \le 1 + \frac{M}{\sqrt{N}} \Big] \le (1-\delta)^{\sqrt{N}}. \qquad (16)$$

The basic inequality for the rest of (3) requires estimating from above the probability that any eigenvalue lies outside the disk of radius $1 + \alpha_N$. We have that

$$1 - \mathcal{P}_N \Big[R_N \le 1 + \alpha_N \Big] \le 1 - \prod_{k=0}^{N-1} \Big(1 - \mathbf{P}\Big[\frac{1}{N} \sum_{\ell=0}^{k} X_\ell \ge 1 + \alpha_N \Big] \Big)$$

$$
\begin{aligned}
&\leq \; 1-\Big[1-\exp(-N\alpha_N+N\log(1+\alpha_N))\Big]^N \\
&\leq \; 1-\Big[1-\exp(-\frac{1}{2}N\alpha_N^2)\Big]^N \leq 1-\exp\,[-\frac{1}{N^{1+\delta/2}}] \leq \frac{1}{N^{1+\delta/2}}, \qquad (17)
\end{aligned}
$$

holding for the choice $\alpha_N = (2+\delta)\sqrt{N^{-1}\log N}$. Given (16) and (17) the proof is completed by an application of the first Borel-Cantelli Lemma. ■

Proof of Corollary 1 For Gaussian quaternion components, Mehta ([11], page 302) contains an exact eigenvalue density $\mathcal{P}_N^Q$ as well as, what is needed here, an expression for averages of test functions of the form $\prod_{k=1}^N h(z_k)$. With things normalized properly,

$$
\begin{aligned}
&\int_{\mathbf{R^2}} \cdots \int_{\mathbf{R^2}} \Big[\prod_{k=1}^N h(x_k,y_k)\Big] d\mathcal{P}_Q^N(dx_1dy_1,\ldots,dx_Ndy_N) \\
&= \; \pi^{-N}\prod_{k=1}^N \frac{N^{2k}}{(2k)!} \times \Big[\det\big[\psi_{ij}(h)\big]_{0\leq i,j\leq 2N-1}\Big]^{\frac{1}{2}}. \qquad (18)
\end{aligned}
$$

where the entries of the determinant are defined by

$$
\psi_{ij}(h) = \int_{\mathbf{R^2}} e^{-2N|z|}(z-\bar{z})h(z)(z^i\bar{z}^j - z^j\bar{z}^i)dxdy.
$$

Now, if $h(z) = 1_{|z|\leq a}$ we find that $\psi_{ij} = \pm\pi\int_0^a e^{-2Nr^2}r^{2j+2}dr$ if $i-j=\mp 1$ and $\psi_{ij}=0$ otherwise. This along with (18) implies that

$$
\mathcal{P}_N^Q\Big(\max_{1\leq k\leq N} z_k \leq a\Big) = \prod_{k=0}^{N-1}\mathbf{P}\Big(\frac{1}{2N}\sum_{\ell=1}^{2(N-k)} X_\ell \leq a^2\Big)
$$

where again X_ℓ are independent exponentials of mean one. From here it is plain that the proofs of Theorem 1 and Theorem 2 may be repeated almost verbatim save for the adjustment of a few constants. ■

Acknowledgments It is a pleasure to thank Stephanos Venakides for helpful discussions. Thanks as well to the referees for several nice suggestions. This work was supported in part by the grant NSF-9983320.

References

[1] Z.D. Bai, Circular Law, *Ann. Probab.* **25**, 1997, 494-529.

[2] Z.D. Bai, J. Silverstein, Exact separation of eigenvalues of large-dimensional sample covariance matrices, *Ann. Probab.* **27**, 1999, 1536-1555.

[3] Z.D. Bai, Y. Yin, Necessary and sufficient conditions for the almost sure convergence of the largest eigenvalue of a Wigner matrix, *Ann. Probab.* **16**, 1988, 1729-1741.

[4] R.N. Bhattacharya, R. Ranga Rao, Normal Approximation and Asymptotic Expansions, Wiley, New York, 1976.

[5] A. Edelman The probability that a random real Gaussian matrix has k real eigenvalues, related distributions, and the circular law, *J. Multivariate Anal.* **60** 1997, 203-232.

[6] P.J. Forrester, G. Honner, Exact statistical properties of the zeroes of complex Gaussian random polynomials, *J. Phys. A: Math. Gen.* **32** 1999, 2961-2981.

[7] S. Geman, The spectral radius of large random matrices, *Ann. Probab.* **14** 1986, 1318-1328.

[8] J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices, *J. Mathematical Phys.*, **6**, 1965. 440-449.

[9] C-R Hwang, A brief survey on the spectral distribution of large random matrices with i.i.d. entries, *Contemp. Math.* **50** AMS, Providence. 1986. 145-152.

[10] I. Johnstone, On the distribution of the largest eigenvalue in principal components analysis, *Ann. Statist*, **29**, 2001. 295-327.

[11] M. L. Mehta, Random Matrices (second edition), Academic Press, Boston, 1991.

[12] A. Soshnikov, Universality at the edge of the spectrum in Wigner random matrices, *Communications in Math Phys.*, **207**, 1998, 697-733.

[13] C. Tracy, H. Widom, Level-spacing distributions and the Airy kernel, *Communications in Math Phys.*, **159**, 1994, 151-174.

[14] C. Tracy, H. Widom, On orthogonal and symplectic matrix ensembles, *Communications in Math Phys.*, **177**, 1996, 727-754.

[15] Y.V. Fyodorov, H.J. Sommers, B. Khoruzhenko, Universality in the random matrix spectra of weak non-Hermiticity, *Ann L'Institut Henri Poincare - Phys theory* **68** no.4 2000, 449-488.