Parabolic Behavior of a Hyperbolic Delay Equation

Thomas Laurent, B. Rider, Michael Reed[1]

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Abstract It is shown that the fundamental solution of a hyperbolic partial differential equation with time delay has a natural probabilistic structure, i.e. is approximately Gaussian, as $t \to \infty$. The proof uses ideas from the DeMoivre proof of the Central Limit Theorem. It follows that solutions of the hyperbolic equation look approximately like solutions of a diffusion equation with constant convection as $t \to \infty$.

1. Introduction

It has long been known that time delays have a smoothing effect on ordinary differential equations. For example, Kolmanovskii and Myshkis [8] state that, "This property of 'solution smoothing'... together with some other properties make retarded differential equations resemble parabolic differential equations. However, the reasons for this resemblance are not entirely clear." In this paper, we provide an analytical foundation for these ideas by studying the initial value problem for the linear hyperbolic equation

$$\frac{\partial}{\partial t}u(t,x) + c\frac{\partial}{\partial x}u(t,x) = -Au(t,x) + Bu(t-\tau,x), \tag{1}$$

$$u(t,x) = f(t,x), \qquad \text{for } -\tau \le t \le 0, \tag{2}$$

where A and B are positive constants and τ is the time delay.

[1]Department of Mathematics, Duke University, Durham, NC 27708

In Section 2, we introduce the 'fundamental solution',

$$K(t,x) \;=\; \sum_{k=0}^{n} \gamma_k(t)\delta_{ct-ck\tau}(x), \quad \text{for } \; n\tau \le t < (n+1)\tau \tag{3}$$

where $n = 0, 1, 2, \ldots$, and we define $K(t,x) \equiv 0$ for $t\epsilon[-\tau, 0)$. The coefficients $\gamma_k(t)$ are defined recursively: $\gamma_0(t)$ satisfies $\gamma_0'(t) = -A\gamma_0(t)$ with $\gamma_0(0) = 1$ and

$$\gamma_k'(t) = -A\gamma_k(t) \;+\; B\gamma_{k-1}(t-\tau), \quad \text{with} \quad \gamma_k(k\tau) = 0. \tag{4}$$

Using standard functional analytic methods, we prove that $K(t) \equiv K(t,\cdot)$ is a continuous $\mathcal{D}'(\mathbb{R})$-valued function on $[0,\infty)$ that is differentiable except at τ. We prove that if $f(t)$ is any continuous $\mathcal{D}'(\mathbb{R})$-valued initial data on $[-\tau, 0]$, then there is a unique $\mathcal{D}'(\mathbb{R})$-valued solution of (1) and it is given by:

$$u(t) \;=\; K(t) * f(0) \;+\; B\int_{-\tau}^{0} K(t-\theta-\tau) * f(\theta)d\theta, \quad \text{for } \; t > 0. \tag{5}$$

If one integrates equation (1) with respect to x (assuming that $u(t,x) \to 0$ as $|x| \to \infty$), one obtains a solution of the retarded differential equation (RDE)

$$y'(t) = -Ay(t) + By(t-\tau), \quad \text{for } \; t > 0. \tag{6}$$

We shall denote the fundamental solution of (6) by $Y(t)$, i.e., $Y(t)$ is the solution with initial data equal to 1 at $t = 0$ and equal to 0 at earlier times. The general solution of equation (6) is given by a formula similar to equation (5) where $Y(t)$ replaces $K(t)$ and convolution is replaced by multiplication, [7].

In Section 3, the main section of the paper, we analyze the asymptotic behavior as $t \to \infty$ of the fundamental solution $K(t)$. The characteristic function of equation (6), is $h(\lambda) = \lambda + A - Be^{-\lambda\tau}$. The root of h with largest real part, λ_0, is real and we define $p = 1/h'(\lambda_0)$ and $\alpha = \tau c^2(1-p)p^2$. We prove that $K(t)$, after normalization, looks asymptotically like a Gaussian with standard deviation $\sigma = \sqrt{\alpha t}$ translating at speed cp. The proof depends on the combinatorics of the functions $\gamma_k(t)$ and follows many of the ideas of the DeMoivre proof of the Central Limit Theorem.

In Section 4, we assume that f is a continuous function on $[-\tau, 0]$ with values in $L^1(\mathbb{R})$. Using an appropriate seminorm, we show that asymptotically the solution $u(t)$ of (1) looks likes a weighted average of solutions of the transport heat equation,

$$\frac{\partial}{\partial t}v(t,x) \;+\; cp\frac{\partial}{\partial x}v(t,x) \;=\; \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}v(t,x), \tag{7}$$

on the space scale $c\tau$.

We note that the equation (1) has a natural interpretation in population dynamics. $u(t, x)$ is the population density in space at time t. Members of the population move to the right at speed c, die at rate A, and produce eggs at a rate B. The eggs are stationary, but after a time τ become moving members of the population. In this language, $Y(t)$ plays the role of the total population at time t. Hyperbolic equations with time delays occur frequently in ecology and cell biology ([3],[9],[11]) where $u(t, x)$ is the density of the population at different maturities x. Often these equations are nonlinear, boundary value problems with inhomogeneous velocities, so it is not clear whether the probabilistic methods of this study can be extended to those cases.

2. Existence, Uniqueness, and Representation

We denote by $\mathcal{D}(\mathbb{R})$ the usual space of test functions of compact support and by $\mathcal{D}'(\mathbb{R})$ the space of distributions. By convergence of a sequence $\mu_n \to \mu$ in $\mathcal{D}'(\mathbb{R})$, we always mean weak convergence, i.e. $< \mu_n, \psi > \to < \mu, \psi >$ for all $\psi \epsilon \mathcal{D}(\mathbb{R})$. For each fixed t, the sum defining $K(t)$ is finite, so it is clear that $K(t) \epsilon \mathcal{D}'(\mathbb{R})$ and the support of $K(t)$ is contained in the interval $[0, ct]$. Since we are interested in the properties of K as a function of t, we begin by summarizing briefly the basic definitions and properties of $\mathcal{D}'(\mathbb{R})$-valued functions that we use repeatedly.

A $\mathcal{D}'(\mathbb{R})$-valued function, f, is said to be *continous* at t_o if $< f(t), \psi >$ is continuous at t_o for all $\psi \epsilon \mathcal{D}(\mathbb{R})$.

Proposition 2.1. If f is continuous at t_o, then:
(a) If $t_n \to t_o$, then $f(t_n) \to f(t_o)$ in $\mathcal{D}'(\mathbb{R})$.
(b) $\partial_x f$ is continuous at t_o.
(c) If $f(t)$ has support in $[0, ct]$ for each t and $v \epsilon \mathcal{D}'(\mathbb{R})$, then $f(t) * v$ is continuous at t_o.

Proof. (a) simply reformulates the definition. Since $< \partial_x f(t), \psi > = < f(t), -\partial_x \psi >$, (b) is immediate. To prove (c), suppose $t_n \to t_o$. Since there is a ball $\mathbb{B}$ that contains the supports of all the distributions $f(t_n)$ and since $f(t_n) \to f(t_o)$ in $\mathcal{D}'(\mathbb{R})$ by assumption, we conclude that $f(t_n) * v \to f(t_o) * v$ using the bi-continuity of convolution on $\mathcal{D}'(\mathbb{R}) \times \mathcal{D}'(\mathbb{R})$([5, p. 105], [6, p. 71]). □

A $\mathcal{D}'(\mathbb{R})$-valued function, f, is said to be *differentiable* at t_o if $< \frac{f(t_n)-f(t_o)}{t_n - t_o}, \psi >$ converges for all $\psi \epsilon \mathcal{D}(\mathbb{R})$ as $t_n \to t_o$.

Proposition 2.2. If f is differentiable at t_o, then:
(a) There exists $f'(t_o)$ in $\mathcal{D}'(\mathbb{R})$ such that $\frac{f(t_n)-f(t_o)}{t_n - t_o} \to f'(t_o)$ in $\mathcal{D}'$.
(b) $\partial_x f$ is differentiable at t_o and $(\partial_x f)'(t_o) = \partial_x f'(t_o)$.

(c) If $f(t)$ has support in $[0, ct]$ for each t and $v\epsilon\mathcal{D}'(\mathbb{R})$, then $f(t) * v$ is differentiable at t_o and $(f(t) * v)' = f'(t) * v$.

Proof. (a) A weakly convergent sequence in $\mathcal{D}'(\mathbb{R})$ has a limit in $\mathcal{D}'(\mathbb{R})$, [10, p. 15]. Since,

$$< \partial_x \frac{f(t_n) - f(t_o)}{t_n - t_o}, \psi > = \; - < \frac{f(t_n) - f(t_o)}{t_n - t_o}, \partial_x \psi >,$$

(b) follows by taking limits. To prove (c), one writes the difference quotient and takes limits using the bi-continuity of convolution as above. □

A $\mathcal{D}'(\mathbb{R})$-valued function, f, is said to be *Riemann integrable* on $[a, b]$ if $< f(t), \psi >$ is Riemann integrable on $[a, b]$ for all $\psi\epsilon\mathcal{D}(\mathbb{R})$.

Proposition 2.3. If f is Riemann integrable on $[a, b]$, then:
(a) There exists an element of $\mathcal{D}'(\mathbb{R})$, denoted $\int_a^b f(t)\, dt$, such that

$$< \int_a^b f(t)\, dt, \psi > = \int_a^b < f(t), \psi > \, dt \qquad \text{for all } \psi \, \epsilon\mathcal{D}(\mathbb{R}).$$

(b) $\partial_x \int_a^b f(t)\, dt \; = \; \int_a^b \partial_x f(t)\, dt$.

Proof. Let $\mathcal{P}_n$ be a sequence of partitions whose mesh size goes to zero as $n \to \infty$. Since $< \sum_{t_i \epsilon \mathcal{P}_n} f(t_i)(t_i - t_{i-1}), \psi >$ converges for each ψ, $\sum_{t_i \epsilon \mathcal{P}_n} f(t_i)(t_i - t_{i-1})$ has a limit in $\mathcal{D}'(\mathbb{R})$. (b) follows, as above, by the adjoint relation for ∂_x. □

The following lemma summarizes the smoothness properties of $K(t)$.

Lemma 2.4. Let $K(t)$ be defined by (3) and (4).
(a) $K(t)$ is continuous on $[0, \infty)$.
(b) $K(t)$ is differentiable on $(0, \infty)\backslash\{\tau\}$ and

$$K'(t) \;\; = \;\; -c\partial_x K(t) \; - \; AK(t) \; + \; BK(t - \tau), \qquad \text{for } t\epsilon(0, \infty)\backslash\{\tau\}. \tag{8}$$

Proof. Each $\delta_{ct-ck\tau}$ is a continuously differentiable $\mathcal{D}'(\mathbb{R})$-valued function of t and the coefficients $\gamma_k(t)$ are C^∞ functions of t since they are the solutions of ordinary differential equations with C^∞ source terms. Thus, $K(t)$ is continuously differentiable on each open interval of the form $(n\tau, (n+1)\tau)$ and we need just check the points $n\tau$ where the definition, (3), changes. Since $\gamma_n(n\tau) = 0$, for $n \geq 1$, $K(t)$ is continuous on $[0, \infty)$. Similarly, (4) shows that $\gamma_n'(n\tau) = 0$ for $n \geq 2$, which implies that $K(t)$ is differentiable at $n\tau$ for $n \geq 2$.

If $t\epsilon(0, \tau)$, then for all $\psi\epsilon\mathcal{D}(\mathbb{R})$, we know that $< K(t), \psi >= \gamma_0(t)\psi(ct)$. Thus,

$$\begin{aligned} \frac{d}{dt} < K(t), \psi > \;\; &= \;\; -A\gamma_0(t)\psi(ct) \; + \; c\gamma_0(t)\psi'(ct) \\ &= \;\; -A < K(t), \psi > \; - \; c < \partial_x K(t), \psi > . \end{aligned}$$

Since $K(t-\tau)=0$, (8) holds.

Let $t \epsilon (n\tau,(n+1)\tau)$ for $n \geq 1$. Then,

$$\begin{aligned}\frac{d}{dt} < K(t),\psi > &= \frac{d}{dt}\sum_{k=0}^{n}\gamma_k(t)\psi(ct-ck\tau)\\ &= -A\gamma_0(t)\psi(ct)+c\gamma_0(t)\psi'(ct)\\ &\quad+\sum_{k=1}^{n}(-A\gamma_k(t)+B\gamma_{k-1}(t-\tau))\psi(ct-ck\tau)\ +\ c\gamma_k(t)\psi'(ct-ck\tau)\\ &= -c < \partial_x K(t),\psi > \ - A < K(t),\psi > \ + B < K(t-\tau),\psi > .\end{aligned}$$

The calculation for $t=(n+1)\tau$ is the same because $\gamma_{n+1}((n+1)\tau)=\gamma'_{n+1}((n+1)\tau)=0$. Thus, (8) holds on $(n\tau,(n+1)\tau]$ for all $n \geq 1$. □

Lemma 2.5. Let $f(t)$ be a continuous $\mathcal{D}'(\mathbb{R})$-valued function on $[-\tau,0]$. Then, the function $t \longmapsto \int_{-\tau}^{0} K(t-\theta-\tau)*f(\theta)\,d\theta$ is well-defined and differentiable on $(0,\infty)$ and:

$$\text{for } t \geq \tau,\quad \frac{d}{dt}\int_{-\tau}^{0} K(t-\theta-\tau)*f(\theta)\,d\theta = \int_{-\tau}^{0} K'(t-\theta-\tau)*f(\theta)\,d\theta, \tag{9}$$

$$\text{for } t\epsilon(0,\tau),\quad \frac{d}{dt}\int_{-\tau}^{0} K(t-\theta-\tau)*f(\theta)\,d\theta = f(t-\tau)+\int_{-\tau}^{0} K'(t-\theta-\tau)*f(\theta)\,d\theta. \tag{10}$$

Proof. For $n=-1,0,1,2,\ldots$, define the following regions of the $t-\theta$ plane.

$$R_n = \{(t,\theta)\mid n\tau \leq t-\theta-\tau < (n+1)\tau \quad \text{and} \quad -\tau \leq \theta \leq 0\}.$$

Since $K(t)$ is a C^∞ function of t in the open intervals $(n\tau,(n+1)\tau)$, $K(t-\theta-\tau)$ is C^∞ except on the boundaries of the regions R_n indicated by the dashed lines in Figure 1. We suppose $\psi\epsilon\mathcal{D}$ and let T_a be the translation operator $T_a\psi(x)=\psi(x-a)$. For (t,θ) in $\bigcup_{-1}^{\infty} R_n$, we define

$$z(t,\theta) \equiv < K(t-\theta-\tau)*f(\theta),\psi >,$$

and note that $z\equiv 0$ on R_{-1} since $K(t)=0$ for $t<0$. On R_0, we have

$$z(t,\theta) = < \gamma_0(t-\theta-\tau)\delta_{c(t-\theta-\tau)}*f(\theta),\psi > = \gamma_0(t-\theta-\tau) < f(\theta),T_{-c(t-\theta-\tau)}\psi > .$$

Note that if $a_n \to a$, then $T_{a_n}\psi \to T_a\psi$ in $\mathcal{D}$, so by the continuity of f and the bicontinuity of $<\cdot,\cdot>$ on $\mathcal{D}'\times\mathcal{D}$, we conclude that z is continuous on R_0. Since this is true for all $\psi\epsilon\mathcal{D}$, Proposition 2.3 assures us that for $0<t<\tau$ the integral $\int_{-\tau}^{0} K(t-\theta-\tau)*f(\theta)\,d\theta$ makes sense in $\mathcal{D}'$.

Let R_0^o denote the interior of R_0. If $(t,\theta)\epsilon R_0^o$ then, $0<t-\theta-\tau<\tau$, so z is differentiable. Using the general fact ([10, p. 105]) that $\frac{d}{dy} < \mu(x),\psi(x+y) > = < \mu(x),\psi'(x+y) >$, we find:

$$\frac{\partial}{\partial t} z(t,\theta) \;=\; \gamma_0'(t-\theta-\tau) < f(\theta), T_{-c(t-\theta-\tau)}\psi > \; + c\gamma_0(t-\theta-\tau) < f(\theta), T_{-c(t-\theta-\tau)}\psi' > .$$

This formula shows that $\frac{\partial}{\partial t}z(t,\theta)$ is continuous on R_0^o and can be extended continuously to the boundary of R_0^o. Thus, $\frac{\partial}{\partial t}z(t,\theta)$ is bounded in R_0^o and the usual proof (using the mean value theorem and the dominated convergence theorem) shows that:

$$\frac{d}{dt}\int_{-\tau}^{0} z(t,\theta)\,d\theta \;=\; \frac{d}{dt}\int_{-\tau}^{t-\tau} z(t,\theta)\,d\theta \;=\; z(t,t-\tau) \;+\; \int_{-\tau}^{t-\tau} \frac{\partial}{\partial t} z(t,\theta)\,d\theta \qquad \text{for } 0<t<\tau.$$

Since this is true for all $\psi \epsilon \mathcal{D}$, Proposition 2.3 guarantees that (10) holds in $\mathcal{D}'$.

Formula (9) is proven similarly. The proof relies on the fact that z is continuous and $\frac{\partial}{\partial t}z(t,\theta)$ is bounded on $\bigcup_0^\infty R_n^o$. □

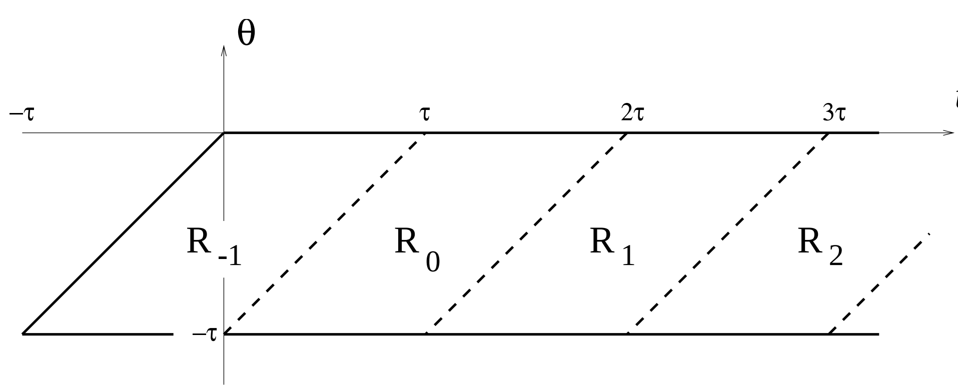

Figure 1: The regions R_n.

Theorem 2.6. Let $f : [-\tau, 0] \to \mathcal{D}'$ be continuous and define:

$$u(t) = \begin{cases} K(t) * f(0) \;+\; B\int_{-\tau}^{0} K(t-\theta-\tau) * f(\theta)\,d\theta, & \text{if } t>0, \\ f(t), & \text{if } -\tau \le t \le 0. \end{cases} \qquad (11)$$

Then, $u(t)$ is the unique $\mathcal{D}'$-valued function that is continuous on $[-\tau,\infty)$, differentiable on $(0,\infty)$, satisfies $u(t) = f(t)$ for $-\tau \le t \le 0$, and

$$u'(t) \;=\; -c\partial_x u(t) \;-\; Au(t) \;+\; Bu(t-\tau) \qquad \text{for } t>0. \qquad (12)$$

Proof. Using the technical details were covered in the propositions and Lemma 2.5, the existence part of the proof is straightforward. For $t \geq \tau$,

$$\begin{aligned} u'(t) &= K'(t) * f(0) \;+\; B\int_{-\tau}^{0} K'(t-\theta-\tau) * f(\theta)\, d\theta \\ &= \{-c\partial_x K(t) - Ak(t) + BK(t-\tau)\} * f(0) \\ &\quad + B\int_{-\tau}^{0} \{-c\partial_x K(t-\theta-\tau) - AK(t-\theta-\tau) + BK(t-\theta-2\tau)\} * f(\theta)\, d\theta \\ &= -c\partial_x u(t) \;-\; Au(t) \;+\; Bu(t-\tau), \end{aligned}$$

and for $0 < t < \tau$,

$$\begin{aligned} u'(t) &= K'(t) * f(0) \;+\; Bf(t-\tau) \;+\; B\int_{-\tau}^{t-\tau} K'(t-\theta-\tau) * f(\theta)\, d\theta \\ &= \{-c\partial_x K(t) - AK(t)\} * f(0) \;+\; Bu(t-\tau) \\ &\quad + B\int_{-\tau}^{t-\tau} \{-c\partial_x K(t-\theta-\tau) - AK(t-\theta-\tau)\} * f(\theta)\, d\theta \\ &= -c\partial_x u(t) \;-\; Au(t) \;+\; Bu(t-\tau). \end{aligned}$$

To prove uniqueness, we need just show that $u(t) \equiv 0$ if $u(t)$ is a differentiable $\mathcal{D}'$-valued function on $(0,\infty)$ that satisfies (12) and $u(t) = 0$ for $-\tau \leq t \leq 0$. Let $\psi \epsilon \mathcal{D}$ and consider the function $< u(t), T_{ct}\psi >$. By writing down the difference question and taking the limit (using the bi-continuity of $< \cdot, \cdot >$ on $\mathcal{D}' \times \mathcal{D}$), one easily sees using (12) that $< u(t), T_{ct}\psi >$ is differentiable on $(0,\tau]$ and

$$\begin{aligned} \frac{d}{dt} < u(t), T_{ct}\psi > &= < u'(t) + c\partial_x u(t), T_{ct}\psi > && (13) \\ &= < -Au(t) + Bu(t-\tau), T_{ct}\psi > && (14) \\ &= -A < u(t), T_{ct}\psi > . && (15) \end{aligned}$$

Thus, for any $h > 0$, we have $< u(t), T_{ct}\psi >= e^{-A(t-h)} < u(h), T_{ch}\psi >$. Taking the limit as $h \to 0$ and using the continuity of u we conclude that $< u(t), T_{ct}\psi >= 0$ for $0 \leq t \leq \tau$. Since this is true for all ψ, we have that $u(t) = 0$ in $\mathcal{D}'$ for $0 \leq t \leq \tau$. By iterating this argument, we find that $u(t) = 0$ for all $t > 0$. □

We remark that if the initial data, f, is a C^1 function on the strip $\mathbb{R} \times [-\tau, 0]$, the distribution solution is C^1 for $t > 0$ and satisfies the differential equation in the classical sense. Similarly, if f is a continuous function of t with values in $L^1(\mathbb{R})$ on $[-\tau, 0]$, then the solution $u(t)$ will be in $L^1(\mathbb{R})$ for all $t > 0$. These theorems can easily be proven by rewriting the differential equation as an integral equation and using the contraction mapping principle. Finally, consider the special case where $f(\theta) = 0$ for $\theta < 0$ and $f(0)$ is a non-negative L^1 function. Then $u(t, \cdot)$ is the convolution of a non-negative function with a finite linear combination of delta functions with positive coefficients, and so $u(t,\cdot)$ is non-negative. Thus, integration in x shows that

$$||u(t,\cdot)||_1 \;=\; Y(t)||f(0)||_1,$$

where $Y(t)$ is the fundamental solution of (6).

3. The Asymptotic Behavior of K

The fundamental solution $K(t)$, which we denote now by K_t, is for each t a finite sum of point masses with weights $\gamma_k(t)$. K_t may be normalized to produce a proper probability measure, $\Pi_t = K_t/K_t(\mathbb{R})$ where $K_t(\mathbb{R}) = \sum_{k=0}^{n} \gamma_k(t)$, for $n\tau \le t < (n+1)\tau$. Note that $K_t(\mathbb{R}) = Y(t)$. The first form of our asymptotic result is stated in the language of convergence in distribution.

Theorem 3.1. Let $h(\lambda)$ be the characteristic polynomial of (6) and λ_0 be the root with the largest real part. Define $p = 1/h'(\lambda_0)$, $q = 1-p$, and $\alpha = c^2\tau p^2 q$. Then,

$$\lim_{t\to\infty} \Pi_t\Big[cpt + \sqrt{\alpha t}\; a,\; cpt + \sqrt{\alpha t}\; b\Big] = \int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

for any fixed $-\infty \le a < b \le \infty$.

From a dynamical perspective, Theorem 3.1 says that, normalized to have unit mass, the fundamental solution of the equation with delay τ and speed c resembles $G_t(x) = e^{-(x-cpt)^2/2\alpha t}/\sqrt{2\pi\alpha t}$ in the sense of measures. Theorem 3.1 follows easily from a stronger result of local limit type, Theorem 3.2 below.

Consider the special sequence of times $t_n = n\tau$, $n \to \infty$. For such t_n,

$$K_{t_n} \equiv \sum_{k\in\mathbb{Z}} b_n(k)\delta_{kc\tau}$$

where the $b_n(k)$ have the following form obtained by explicitly solving the equations (4):

$$b_n(k) = \begin{cases} \frac{k^{n-k}}{(n-k)!}\xi^{-k}(B\tau)^n & \text{for } 1 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

where $\xi = B\tau e^{A\tau}$. We let S_n be the total mass of K_{t_n}, *i.e.*,

$$S_n \;=\; \sum_{k=1}^{n} b_n(k) \;=\; Y(t_n). \tag{16}$$

In these terms, Π_{t_n} can be be written

$$\Pi_{t_n} = \sum_{k\in Z} \frac{b_n(k)}{S_n}\delta_{kc\tau}. \tag{17}$$

The following local theorem is the main result of this section. By local we mean that we have uniform control of the individual masses $b_n(\cdot)$.

Theorem 3.2 Let $\pi_n(\mathbb{R})$ be the lattice $\{\frac{k}{\sqrt{n}}\}$ for k ranging over the integers, and let $[pn]$ denote the integer part of pn. Then,

$$\sup_{x\in\pi_n(\mathbb{R})}\left|\sqrt{n}\frac{b_n([pn]+x\sqrt{n})}{S_n} - \frac{1}{\sqrt{2\pi p^2 q}}e^{-\frac{x^2}{2p^2q}}\right| = O\Big(\sqrt{\frac{\log n}{n}}\Big) \tag{18}$$

as $n \to \infty$.

As already mentioned, the proof of Theorem 3.2 is similar in spirit to the classical Demoivre-Laplace calculation leading to the (local) Central Limit Theorem for the Binomial distribution (see, for example [4, Chapter VII, Section 3]). Note that the error on the right hand side of (18) compares favorably with known estimates concerning Gaussian convergence of general lattice distributions (see [1, Chapter 5]).

We break the proof of Theorem 3.2 into several steps. First, we characterize p in a different way. Then, in Lemma 3.4 we describe the shape of the distribution $b_n(\cdot)$ and show that the mass peaks at a sequence $\{m_n\}$ that is within a fixed constant of $\{np\}$, the asymptotic mean. Lemma 3.5 contains the main asymptotic estimate; we renormalize by the mass at the maximum, define

$$f_n(x) \equiv \frac{b_n(m_n + x\sqrt{n})}{b_n(m_n)}, \tag{19}$$

and the heart of the proof is a set of estimates on the sequence $\{f_n(x)\}$. Finally, using the asymptotic estimate, we prove Theorem 3.2 and sketch the proof of Theorem 3.1.

Lemma 3.3. For

$$\psi(x) \equiv \frac{1-x}{x}e^{\frac{1-x}{x}} \quad \text{and} \quad \xi \equiv B\tau e^{A\tau},$$

p is the unique solution of $\xi = \psi(x)$ lying in $(0, 1)$.

Proof. Previously, p was defined as $1/h'(\lambda_0)$ where $h(\lambda) = \lambda + A - Be^{-\lambda\tau}$ and λ_0 is the unique positive root of h. It follows that $(1-p)/p = h'(\lambda_0) - 1$ in which the right hand side may be expressed as either $\tau(\lambda_0 + A)$ or $B\tau e^{-\lambda_0\tau}$. And so,

$$\frac{1-p}{p} = B\tau e^{-\lambda_0\tau} = B\tau e^{A\tau} e^{-\tau(\lambda_0+A)} = \xi e^{-(1-p)/p},$$

as desired. For the uniqueness, observe that $\psi(x) < 0$ for $x < 0$ and $x > 1$ and is strictly decreasing on $[0,1]$ from $+\infty$ at $x = 0$ to 0 at $x = 1$. □

Lemma 3.4. For each n, the sequence $\{b_n(k)\}$ attains its maximum at a single index $= m_n$ and is increasing (decreasing) to the left (right) of that point, respectively. Furthermore, there exists a constant D such that $|m_n - np| \leq D$.

Proof. Define the following approximates to ψ:

$$\psi_n(x) \equiv \Big(\frac{1-x}{x+1/n}\Big)\Big(1 + \frac{1}{nx}\Big)^{n(1-x)}.$$

It may be verified that each ψ_n lies under ψ and that $\lim_{n\to\infty} \psi_n = \psi$ holds pointwise. Moreover, the ψ_n have the same shape as ψ in that they decrease strictly from $+\infty$ to 0 as x ranges from 0 to 1. As such, there is a unique $p_n \in (0,1)$ satisfying $\xi = \psi_n(p_n)$.

For $k = 1, 2, \ldots n-1$,

$$\frac{b_n(k+1)}{b_n(k)} = \frac{1}{\xi}\Big(1 + \frac{1}{k}\Big)^{n-k}\Big(\frac{n-k}{k+1}\Big) = \frac{1}{\xi}\psi_n\Big(\frac{k}{n}\Big),$$

and so, $b_n(k)$ is increasing on $k \leq np_n$ and then decreasing on $k \geq np_n$. If we let $m_n = [p_n n]$, the smallest integer $\geq p_n n$, it is left to prove that $p - p_n = O(1/n)$.

First note that the properties of ψ_n and ψ imply that $p_n < p$ and $\lim_{n\to\infty} p_n = p$. The first follows from $\psi(p) = \xi = \psi_n(p_n) < \psi(p_n)$ along with the strict decrease of ψ; the second follows from the first and the convergence $\psi_n \to \psi$. Now, set $q_n = 1 - p_n$. Since $\xi = \psi_n(p_n)$,

$$-\log\xi + nq_n \log\Big(1 + \frac{1}{np_n}\Big) + \log\Big(\frac{q_n}{p_n + 1/n}\Big) = 0,$$

and using the fact that $p_n \geq p/2$ for all large enough n, an expansion yields

$$-\log\xi + \frac{q_n}{p_n} + \log\Big(\frac{q_n}{p_n}\Big) = O\Big(\frac{1}{n}\Big). \tag{20}$$

Adding $\log\xi = \log\psi(p) = \log(q/p) + (q/p)$ to (20) gives

$$\frac{p-p_n}{pp_n} + \log\Big(\frac{q_n}{q}\frac{p}{p_n}\Big) = O\Big(\frac{1}{n}\Big).$$

The fact that $p > p_n$ (and so $q < q_n$) implies that each term on the left hand side is strictly positive. Finally, $pp_n = O(1)$, showing that $p - p_n = O(\frac{1}{n})$ and the proof is complete. □

Lemma 3.5. There exists a constant C such that

$$\sup_{x\in\pi_n(\mathbb{R})}\Big|f_n(x) - e^{-\frac{x^2}{2p^2q}}\Big| \le C\frac{1}{\sqrt{n}}. \tag{21}$$

Proof. From the definition of $b_n(k)$,

$$b_n(m_n+\sqrt{n}x) = \frac{(m_n+\sqrt{n}x)^{n-m_n-\sqrt{n}x}}{(n-m_n-\sqrt{n}x)!}\xi^{-(m_n+\sqrt{n}x)}(B\tau)^n$$

for $x \in \pi_n([\frac{1-m_n}{\sqrt{n}}, \frac{n-m_n}{\sqrt{n}}])$. In this range, after dividing by $b_n(m_n)$ and taking the case $x > 0$ we have

$$\begin{aligned} f_n(x) &= \Big(1+\frac{\sqrt{n}x}{m_n}\Big)^{n-m_n}\frac{(n-m_n)!}{(n-m_n-\sqrt{n}x)!}\frac{1}{[\xi(m_n+\sqrt{n}x)]^{\sqrt{n}x}} \\ &= \Big[\Big(1+\frac{\sqrt{n}x}{m_n}\Big)^{n-m_n}e^{-\sqrt{n}x\frac{q}{p}}\Big]\Big[\prod_{0\le k\le\sqrt{n}x-1}\frac{n-m_n-k}{nq}\Big]\Big[\frac{np}{(m_n+\sqrt{n}x)}\Big]^{\sqrt{n}x} \\ &\equiv u_n(x)\times v_n(x)\times w_n(x). \end{aligned} \tag{22}$$

The second line follows after multiplying through by $e^{-\sqrt{n}x\frac{q}{p}}\,(\xi\frac{p}{q})^{\sqrt{n}x} = 1$ and some algebra. If $x < 0$ is desired, $v_n(x)$ should be replaced by $\prod_{1\le k\le\sqrt{n}|x|}(\frac{nq}{n-m_n+k})$. However, the asymptotic considerations are similar for either $x > 0$ or $x < 0$; we assume $x > 0$ for the rest of the proof. We note that $f_n(x) = 0$ for x outside the interval $[\frac{1-m_n}{\sqrt{n}}, \frac{n-m_n}{\sqrt{n}}]$, so we define u_n, v_n and w_n to be zero there too.

We first estimate the quantities

$$|u_n(x) - e^{-\frac{qx^2}{2p^2}}|, |v_n(x) - e^{-\frac{x^2}{2q}}|, \text{ and } |w_n(x) - e^{-\frac{x^2}{p}}|$$

on the interval $0 \le x \le M\sqrt{\log n}$ for suitable M. This interval is convenient for the tail estimates below. Using the elementary inequalities $a - a^2 \le \log(1+a) \le a + a^2$ for $|a| \le 1/2$, on the range $0 \le x \le \frac{p}{4} n^{\frac{1}{2}}$ we have that

$$\begin{aligned}\log w_n(x) &\le -\sqrt{n}x \log\Big(1 + \frac{x}{\sqrt{n}p} - \frac{D}{n}\Big) \\ &\le -\sqrt{n}x\Big(\frac{x}{\sqrt{n}p} - \frac{D}{n}\Big) + 2\sqrt{n}x\Big((\frac{x}{\sqrt{n}p})^2 + (\frac{D}{n})^2\Big) \le -\frac{x^2}{p} + 2\Big(\frac{Dx}{\sqrt{n}} + \frac{x^3}{\sqrt{n}p^2}\Big)\end{aligned}$$

and,

$$\log w_n(x) \ge -\frac{x^2}{p} - 2\Big(\frac{Dx}{\sqrt{n}} + \frac{x^3}{\sqrt{n}p^2}\Big),$$

for all large n. It follows that there exist constants c_1, c_2, c_3, so that

$$|w_n(x) - e^{-x^2/p}| \le |e^{-x^2/p}(e^{\frac{c_1}{\sqrt{n}}(1+x^3)} - 1)| \le \frac{c_2}{\sqrt{n}}(1 + |x|^3)e^{-x^2/p} \le \frac{c_3}{\sqrt{n}} \tag{23}$$

for all x in $[0, n^{\frac{1}{6}}]$. Of course, for n large $[0, M\sqrt{\log n}]$ is contained in both $[0, \frac{p}{4}n^{\frac{1}{2}}]$ and $[0, n^{\frac{1}{6}}]$.

For $v_n(x)$,

$$\begin{aligned}\log v_n(x) &\le \sum_{0 \le k \le \sqrt{n}x - 1} \log\Big(1 - \frac{k - D}{nq}\Big) \qquad (24) \\ &\le \sum_{0 \le k \le \sqrt{n}x - 1} \Big(\frac{-k}{nq} + \frac{2k^2}{n^2q^2}\Big) + 2\frac{Dx}{\sqrt{n}} \le -\frac{x^2}{2q} + \frac{2}{\sqrt{n}}\Big(1 + Dx + \frac{x^3}{q^2}\Big),\end{aligned}$$

with a lower bound of the form

$$\log v_n(x) \ge -\frac{x^2}{2q} - \frac{2}{\sqrt{n}}\Big(1 + Dx + \frac{x^3}{q^2}\Big). \tag{25}$$

For $u_n(x)$, one goes out to third order to find:

$$-\frac{qx^2}{2p^2} - \frac{q}{\sqrt{n}p}(2Dx + x^2 + 8x^3) \le \log u_n(x) \le -\frac{qx^2}{2p^2} + \frac{q}{\sqrt{n}p}(2Dx + x^2 + 8x^3) \tag{26}$$

Given the inequalities, (24), (25) and (26), an argument similar to that for $w_n(x)$ shows that both $|v_n(x) - e^{-x^2/2q}|$ and $|u_n(x) - e^{-qx^2/2p^2}|$ are $O(1/\sqrt{n})$ for $x \in [0, M\sqrt{\log n}]$. It follows that the difference between $f_n(x)$ and $e^{-x^2/2p^2q} = e^{-x^2/p}e^{-x^2/2q}e^{-qx^2/2p^2}$ is also of order $1/\sqrt{n}$ for those values of x.

Finally, we prove a global estimate

$$f_n(x) \le c_4 e^{-c_5 x^2} \equiv f^*(x), \quad \text{for } x \epsilon \mathbb{R}, \tag{27}$$

which will give us control over the tail region $x \geq M\sqrt{\log n}$. As above, we give the proof only for $x > 0$. First,

$$\log u_n(x) = (n - m_n) \log\Big(1 + \frac{\sqrt{n}x}{m_n}\Big) - \frac{\sqrt{n}qx}{p} \leq \Big(\frac{\sqrt{n}x}{m_n}\Big)\Big(\frac{np - m_n}{p}\Big) \leq \frac{2D}{p^2},$$

and the product defining $v_n(x)$ has only a finite number of terms greater than one independent of n. Thus, $|u_n v_n|$ is bounded by a fixed constant for all n and x. To handle w_n, first notice that if n is large enough and $x \geq 2q\sqrt{n}$, then

$$\frac{n - m_n}{\sqrt{n}} \ \leq\ q\sqrt{n} + \frac{D}{\sqrt{n}} \ \leq\ x,$$

so $w_n(x) = 0$ by definition. Also, for n large enough and any $x > 1$, we know that $\frac{x}{\sqrt{n}p} - \frac{D}{n} > 0$, so, there exists a constant, c_6, such that

$$\log\big(1 + \frac{x}{\sqrt{n}p} - \frac{D}{n}\big) \ \geq\ c_6\big(\frac{x}{\sqrt{n}p} - \frac{D}{n}\big), \quad \text{for } \ 1 \leq x \leq 2q\sqrt{n}.$$

It follows that

$$w_n(x) \ \ \leq\ \ e^{-c_6 x^2/p} e^{2c_6 D^2 q} \qquad \text{for } x \geq 1 \ \text{ and large } n. \tag{28}$$

By adjusting the multiplicative constant, this Gaussian bound extends to all of $x > 0$. The bound (27) follows from boundedness of $u_n v_n$ and (28).

To obtain the tail estimate, we choose M so that $M^2 \min\{c_5, \frac{1}{2p^2q}\} \geq \frac{1}{2}$, which implies that both $f_n(x)$ and $e^{-x^2/2p^2q}$ will be of order $1/\sqrt{n}$ on $x > M\sqrt{\log n}$. This concludes the proof. □

Proof of Theorem 3.2. We denote $f(x) = e^{-x^2/2p^2q}$ and define

$$\tilde{f}_n(x) \equiv \frac{b_n([np] + \sqrt{n}x)}{b_n(m_n)}.$$

Since, $\tilde{f}_n(x) = f_n(x - \frac{1}{\sqrt{n}}(m_n - [np]))$, Lemma 3.5 and the triangle inequality imply

$$\Big|\tilde{f}_n(x) - f(x)\Big| \leq C\frac{1}{\sqrt{n}} + \sup_{|\delta| \leq D/\sqrt{n}} \Big|f(x) - f(x + \delta)\Big| \leq C_1 \frac{1}{\sqrt{n}}. \tag{29}$$

Secondly,

$$\begin{aligned}
\sqrt{2\pi p^2 q} - \frac{S_n}{\sqrt{n} b_n(m_n)} &= \int_{-\infty}^{\infty} f(x)dx - \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{n}} f_n(k/\sqrt{n}) \\
&= \Big[\int_{-\infty}^{\infty} f(x)dx - \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{n}} f(k/\sqrt{n})\Big] + \frac{1}{\sqrt{n}}\Big[\sum_{k=-\infty}^{\infty} (f(k/\sqrt{n}) - f_n(k/\sqrt{n}))\Big].
\end{aligned}$$

Here the first term on the right hand side is controlled by the error in the Riemann sum, which is $(1/\sqrt{n}) \int_{-\infty}^{\infty} |f'|$. For the second term, recall that we know that $|f_n(\cdot) - f(\cdot)| \leq C/\sqrt{n}$ and also the bound (27). Therefore,

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} (f(k/\sqrt{n}) - f_n(k/\sqrt{n})) &\leq \sum_{|k| \leq \sqrt{\ell_n n}} \Big[\frac{C}{\sqrt{n}}\Big] + \sum_{|k| \geq \sqrt{\ell_n n}} (c_4 e^{-c_5 k^2/n} + e^{-k^2/2np^2 q}) \\
&\leq 2C\sqrt{\ell_n} + c_6 \sqrt{\frac{n}{\ell_n}} e^{-c_5 \ell_n},
\end{aligned}$$

and taking ℓ_n an appropriate multiple of $\log n$ shows that

$$\Big|\frac{S_n}{\sqrt{n} b_n(m_n)} - \sqrt{2\pi p^2 q}\Big| \leq C_2 \sqrt{\frac{\log n}{n}}. \tag{30}$$

Putting together (29) and (30) completes the proof. □

Finally, we conclude this section with:

Proof of Theorem 3.1 First consider $t \to \infty$ along the special sequence $t_n = n\tau$. For any $a < b$, define

$$I_n(a,b) = \Big[cp(n\tau) + \sqrt{\alpha(n\tau)}\ a, cp(n\tau) + \sqrt{\alpha(n\tau)}\ b\Big] = c\tau\Big[np + \sqrt{p^2 qn}\ a, np + \sqrt{p^2 qn}\ b\Big].$$

Noting that the individual masses of Π_{t_n} are positioned a distance $c\tau$ apart from one and other, we see that

$$\begin{aligned}
\lim_{n\to\infty} \Pi_{t_n}\Big[I_n(a,b)\Big] &= \lim_{n\to\infty} \sum_{\sqrt{p^2 qn} a \leq k - np \leq \sqrt{p^2 qn} b} \Big(\frac{b_n(k)}{S_n}\Big) \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{x \in \pi_n[\sqrt{p^2 q} a, \sqrt{p^2 q} b]} \Big(\sqrt{n} \frac{b_n([np] + \sqrt{n} x)}{S_n}\Big) \\
&= \int_{-\sqrt{p^2 q} a}^{\sqrt{p^2 q} b} e^{-x^2/2p^2 q} \frac{dx}{\sqrt{2\pi p^2 q}}
\end{aligned}$$

by the dominated convergence theorem. The pointwise convergence of the integrand is the statement of the Theorem 3.2, and the domination is (27). This proves the result for the special sequence $t_n = n\tau$. The general statement as $t \to \infty$ follows by interpolation and the triangle inequality. $\square$

4. Comparison to the Transport Heat Equation

In Theorem 3.1 we showed that normalized $\Pi_t = K_t/Y(t)$ looks more and more like

$$G_t(dx) = \frac{e^{-(x-cpt)^2/2\alpha t}}{\sqrt{2\pi\alpha t}}\, dx$$

in the sense of probability measures as $t \to \infty$. This suggests that solutions of the hyperbolic equation with time delay (1) may look like solutions of the transport heat equation (7) for t large. We will prove two theorems that express this idea. For the first, Theorem 4.2, we shall consider the special case in which the initial data, f, is zero except at $t = 0$ and $f(0) \epsilon L^1(\mathbb{R})$. General initial condition are considered in Theorem 4.3.

For $f \epsilon L^1(\mathbb{R})$, define $U(t)$ and $V(t)$ by

$$U(t)f \;\equiv\; K_t * f \quad \text{and} \quad V(t)f \;\equiv\; G_t * f.$$

$U(t)$ is a strongly continuous family of bounded operators and $V(t)$ is a strongly continuous semi-group on $L^1(\mathbb{R})$ for $t \geq 0$; $U(t)f$ satisfies (1) and $V(t)f$ satisfies (7). Furthermore, $||U(t)f||_1 \leq Y(t)||f||_1$ and $||V(t)f||_1 \leq ||f||_1$, with equality in both cases if f is non-negative. We shall see that $U(t)/Y(t)$ and $V(t)$ are "comparable" for large t on the space scale $c\tau$.

Since $K_t/Y(t)$ is, for each t, a finite sum of point measures spaced at intervals of length $c\tau$ and G_t is smooth, we need a method of comparison that integrates over intervals of length $c\tau$. Let $\mathbb{M}(\mathbb{R})$ denote the finite Borel measures on $\mathbb{R}$, and for $\mu \epsilon \mathbb{M}(\mathbb{R})$ define

$$||\mu||_{1,c\tau} \;\equiv\; \sup_{-c\tau \leq a \leq 0} \sum_{k \epsilon \mathbb{Z}} |\mu([a + kc\tau, a + (k+1)c\tau))|.$$

Of course, any $f \epsilon L^1(\mathbb{R})$ corresponds to a finite Borel measure and in that case

$$||f||_{1,c\tau} \;\equiv\; \sup_{-c\tau \leq a \leq 0} \sum_{k \epsilon \mathbb{Z}} \left| \int_{a+kc\tau}^{a+(k+1)c\tau} f(x)\, dx \right|.$$

We begin by collecting the properties of $|| \cdot ||_{1,c\tau}$.

Proposition 4.1

(a)$||\cdot||_{1,c\tau}$. is a semi-norm on $\mathbb{M}(\mathbb{R})$ that satisfies $||\mu||_{1,c\tau} \leq ||\mu||_{\mathbb{M}(\mathbb{R})}$.

(b) If $\mu\epsilon\mathbb{M}(\mathbb{R})$ and $f\epsilon L^1(\mathbb{R})$, then $||\mu * f||_{1,c\tau} \ \leq \ ||\mu||_{1,c\tau}\, ||f||_1$.

(c) If $g(t)$ is a continuous function on $[a,b]$ with values in $L^1(\mathbb{R})$, then $|| \int_a^b g(t)\, dt||_{1,c\tau} \leq \int_a^b ||g(t)||_{1,c\tau}\, dt$.

Proof. The straightforward proof of (a) is omitted. Since g is a continuous function, (c) is proven by using the sub-linearity of the semi-norm in the standard proof for Riemann integrals. To prove (b),

$$\begin{aligned} ||\mu * f||_{1,c\tau} &= \sup_{-c\tau\leq a\leq 0} \sum_{k\epsilon\mathbb{Z}} |(\mu * f)([a+kc\tau, a+(k+1)c\tau))| \\ &= \sup_{-c\tau\leq a\leq 0} \sum_{k\epsilon\mathbb{Z}} | \int_{\mathbb{R}} \mu([a+kc\tau, a+(k+1)c\tau) - y) f(y)\, dy| \\ &\leq \int_{\mathbb{R}} \{ \sup_{-c\tau\leq a\leq 0} \sum_{k\epsilon\mathbb{Z}} |\mu([a+kc\tau, a+(k+1)c\tau) - y)| \}\, |f(y)|\, dy \\ &= ||\mu||_{1,c\tau}\, ||f||_1, \end{aligned}$$

where we have used $(\mu * f)(A) = \int_{\mathbb{R}} \mu(A-y) f(y) dy$ to obtain the second line (see [2], page 266). □

Theorem 4.2 Suppose $f\epsilon L^1(\mathbb{R})$. Then, there exists a constant C such that for t large,

$$||U(t)f/Y(t) \ - \ V(t)f||_{1,c\tau} \ \leq \ C\frac{\log t}{\sqrt{t}}\, ||f||_1. \tag{31}$$

Proof. According to Proposition 4.1(b),

$$||U(t)f/Y(t) \ - \ V(t)f||_{1,c\tau} \ = \ ||(K_t/Y(t)) * f \ - \ G_t * f||_{1,c\tau} \ \leq \ ||(K_t/Y(t)) \ - \ G_t||_{1,c\tau}\, ||f||_1,$$

so we need just prove

$$||K_t/Y(t) \ - \ G_t||_{1,c\tau} \ \ \leq \ \ C\frac{\log t}{\sqrt{t}} \tag{32}$$

for large t. As in Section 3, we give the details for the special sequence $t_n = n\tau$; the proof for general t follows from the triangle inequality. We set $g(x) = e^{-x^2/2p^2q}/\sqrt{2\pi p^2 q}$ and

$$g_n(x) = \sqrt{n}\frac{b_n([np] + \sqrt{n}x)}{S_n}.$$

First, we rewrite the left hand side of (32) so that we can use the machinery and results of Section 3.

$$
\begin{aligned}
||K_{t_n}/Y(t_n) - G_{t_n}||_{1,c\tau} &= \sup_{a\in(-c\tau,0]} \sum_{k\in\mathbb{Z}} \left| \frac{b_n(k)}{S_n} - \frac{1}{\sqrt{2\pi\alpha n\tau}} \int_{a+kc\tau}^{a+(k+1)c\tau} e^{-(x-cpn\tau)^2/2\alpha n\tau}\,dx \right| \\
&= \sup_{a'\in(-1,0]} \sum_{k\in\mathbb{Z}} \left| \frac{b_n(k)}{S_n} - \frac{1}{\sqrt{n}} \int_{a'+k}^{a'+(k+1)} g\Big(\frac{z-np}{\sqrt{n}}\Big) dz \right| \\
&= \sup_{a''\in(-\frac{1}{\sqrt{n}},0]} \sum_{k\in\mathbb{Z}} \left| \frac{1}{\sqrt{n}} g_n(\frac{k}{\sqrt{n}}) - \int_{a''+\frac{k}{\sqrt{n}}}^{a''+\frac{k+1}{\sqrt{n}}} g\Big(y + \frac{[np]-np}{\sqrt{n}}\Big) dy \right| \\
&\le \frac{1}{\sqrt{n}} \sum_{k\in Z} \Big| g_n(\frac{k}{\sqrt{n}}) - g(\frac{k}{\sqrt{n}}) \Big| \\
&\qquad + \sum_{k\in\mathbb{Z}} \int_{\frac{k}{\sqrt{n}}}^{\frac{k+1}{\sqrt{n}}} \Big(\sup_{a''\in(-\frac{1}{\sqrt{n}},0]} \Big| g(\frac{k}{\sqrt{n}}) - g(y + a'' + \frac{[np]-np}{\sqrt{n}}) \Big| \Big) dy \\
&\equiv \mathcal{A}_n + \mathcal{B}_n.
\end{aligned}
$$

To estimate $\mathcal{A}_n$, recall from Theorem 3.2 that $|g_n(\frac{k}{\sqrt{n}}) - g(\frac{k}{\sqrt{n}})| \le C_1(\log n/n)^{1/2}$ independently of k. Also the bound (27) for f_n translates to $g_n(x) \le c_7 e^{-c_5 x^2}$ since $\sqrt{n} b_n(m_n)/S_n$ approaches a limit (see the proof of Theorem 3.2). It follows that

$$
\mathcal{A}_n \le \frac{1}{\sqrt{n}} \sum_{|k|\le M\sqrt{n\log n}} C_1 \sqrt{\frac{\log n}{n}} + \frac{1}{\sqrt{n}} \sum_{|k|\ge M\sqrt{n\log n}} \Big(c_6 e^{-c_5 k^2/n} + g(k/\sqrt{n}) \Big) = O\Big(\frac{\log n}{\sqrt{n}}\Big)
$$

by the choice of M. The estimate for $\mathcal{B}_n$ follows the same strategy. The function g is globally Lipshitz, and so

$$
\sup_{|c|\le\frac{2}{\sqrt{n}}} \Big| g(\frac{k}{\sqrt{n}}) - g(\frac{k}{\sqrt{n}} + c) \Big| \le C_2 \frac{1}{\sqrt{n}}.
$$

for all k. Using this for $|k| \le M\sqrt{n\log n}$ and the decay of g for larger k, we find

$$
\mathcal{B}_n \le \sum_{|k|\le M\sqrt{n\log n}} \frac{1}{\sqrt{n}} \cdot \frac{C_2}{\sqrt{n}} + \sum_{|k|\ge M\sqrt{n\log n}} \frac{1}{\sqrt{n}} \cdot 2g\Big(\frac{k-2}{\sqrt{n}}\Big) = O\Big(\sqrt{\frac{\log n}{n}}\Big).
$$

This proves (32) and thus completes the proof of (31). □

We now consider the case in which f, the initial data for (1), is a continuous $L^1(\mathbb{R})$-valued function of t for $-\tau \le t \le 0$. Define

$$\begin{aligned} u(t) &= U(t)f(0) + B\int_{-\tau}^{0} U(t-\theta-\tau)f(\theta)\,d\theta \\ v(t) &= Y(t)V(t)f(0) + B\int_{-\tau}^{0} Y(t-\theta-\tau)V(t-\theta-\tau)f(\theta)\,d\theta \\ y(t) &= Y(t)||f(0)||_1 + B\int_{-\tau}^{0} Y(t-\theta-\tau)||f(\theta)||_1\,d\theta \end{aligned}$$

$u(t)$ is, of course, the solution of (1) with initial condition (2). Notice that $v(t)$ is not a solution of (7) but is a weighted average of solutions to (7); it will be clear from the proof why this weighted average is natural. $y(t)$ is the solution of (6) with initial data equal to $||f(t)||_1$ for $-\tau \le t \le 0$.

Theorem 4.3. Suppose that f is a continuous $L^1(\mathbb{R})$-valued function of t for $-\tau \le t \le 0$, and let $u(t)$, $v(t)$, and $y(t)$ be defined as above. Then, there is a constant C_1 so that for t large enough,

$$||u(t) - v(t)||_{1,c\tau} \le C_1 \frac{\log t}{\sqrt{t}} y(t). \tag{33}$$

Proof. We subtract $v(t)$ from $u(t)$ and apply the seminorm $||\cdot||_{1,c\tau}$. Using Proposition 4.1(c) and the estimate (32), we find

$$\begin{aligned} ||u(t) - v(t)||_{1,c\tau} &\le Y(t)C\frac{\log t}{\sqrt{t}}||f(0)||_1 + B\int_{-\tau}^{0} Y(t-\theta-\tau)C\frac{\log(t-\theta-\tau)}{\sqrt{t-\theta-\tau}}||f(\theta)||_1\,d\theta \\ &\le C_1\frac{\log t}{\sqrt{t}}\{Y(t)||f(0)||_1 + B\int_{-\tau}^{0} Y(t-\theta-\tau)||f(\theta)||_1\,d\theta\} \end{aligned}$$

for large t, which proves (33). □

Note that if $f(t)$ is a non-negative function for all $t\epsilon[-\tau, 0]$, then $y(t) = ||u(t)||_1 = ||v(t)||_1$ so (33) estimates the relative error.

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