

# On the Information Rate of MIMO Systems with Finite Rate Channel State Feedback and Power On/Off Strategy

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**Abstract**—This paper quantifies the information rate of multiple-input multiple-output (MIMO) systems with finite rate channel state feedback and power on/off strategy. In power on/off strategy, a beamforming vector (beam) is either turned on (denoted by *on-beam*) with a constant power or turned off. We prove that the ratio of the optimal number of on-beams and the number of antennas converges to a constant for a given signal-to-noise ratio (SNR) when the number of transmit and receive antennas approaches infinity simultaneously and when beamforming is perfect. Based on this result, a near optimal strategy, i.e., power on/off strategy with a constant number of on-beams, is discussed. For such a strategy, we propose the *power efficiency factor* to quantify the effect of imperfect beamforming. A formula is proposed to compute the maximum power efficiency factor achievable given a feedback rate. The information rate of the overall MIMO system can be approximated by combining the asymptotic results and the formula for power efficiency factor. Simulations show that this approximation is accurate for all SNR regimes.

## I. INTRODUCTION

This paper considers multiple-antenna systems, also known as multiple-input multiple-output (MIMO) systems, with finite rate channel state feedback and power on/off strategy. MIMO systems have attracted much interest on account of their high spectral efficiency. When perfect channel state information (CSI) is available at both transmitter and receiver (CSITR), the MIMO channel can be viewed as a set of parallel sub-channels. The transmission power on each sub-channel obeys water filling principle [1]. However, perfect CSI at transmitter may require infinite feedback rate, which is not practical. In this paper, systems with finite rate channel state feedback are discussed. For Rayleigh flat fading channel, it has been proved that the optimal transmission strategy is to use the current feedback to select the covariance matrix of the input Gaussian signal [2]. The covariance matrix can be decomposed to an orthonormal matrix and a diagonal matrix which are called beamforming matrix and power control matrix respectively. Power on/off strategy means that a beamforming vector (beam) is either turned on (denoted by *on-beam*) with a constant power or turned off. As we will show later, power on/off strategy is near optimal for MIMO systems.

There are several works dealing with MIMO systems with channel state feedback. MIMO systems with only one on-beam are studied in [3] and [4], where the beamforming codebook design criterion and performance analysis are derived by geometric arguments in the Grassmann manifold. MIMO systems with multiple on-beams are considered in [5]–[9]. Criteria to select the beamforming matrix are developed in [5] and [6]. The signal-to-noise ratio (SNR) loss due to quantized beamforming is discussed in [7]. The corresponding analysis is only valid for MIMO systems with asymptotically large number of transmit antennas. The effect of beamforming quantization on information rate is investigated in [8] and [9]. The loss in information rate is quantified for high SNR region in [8] where a metric other than the chordal distance is employed. In [9], a formula to calculate the information rate for all SNR regimes is proposed by letting the numbers of transmit antennas, receive antennas and feedback rate approach infinity simultaneously. But the performance approximated by this formula for several feedback bits may go beyond that of CSITR case.

The contributions of this paper are as follows. 1) We provide a framework to design power on/off strategy avoiding assuming a constant number of on-beams initially as in [6]–[9]. We prove that the ratio of the optimal number of on-beams and the number of antennas converges to a constant for a given SNR when the number of transmit and receive antennas approaches infinity simultaneously and when beamforming is perfect. This result suggests a power on/off strategy with a constant number of on-beams. We also derive asymptotic formulas to calculate the optimal number of on-beams and the corresponding information rate. Simulations show that a constant number of on-beams is near optimal for practical MIMO systems. 2) A new method is developed to quantify the effect of beamforming matrix quantization based on a random matrix analysis on the Grassmann manifold. When the number of on-beams is a constant, the corresponding beamforming codebook contains beamforming matrices with the same rank. The *power efficiency factor* is proposed to quantify the effect of such a beamforming codebook. A formula is proposed to compute the maximum power efficiency factor achievable as  $a$

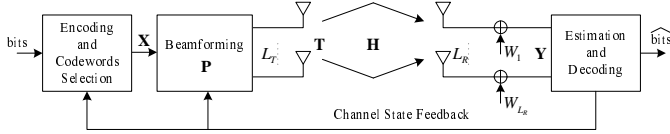


Fig. 1. System model

*function of feedback rate.* Combining the asymptotic formulas for perfect beamforming and the formula for power efficiency factor, the information rate of the overall MIMO system can be accurately approximated for all SNRs.

## II. SYSTEM MODEL AND THE DESIGN PROBLEM

The model of a wireless communication system with  $L_T$  transmit antennas,  $L_R$  receive antennas and finite rate channel state feedback is depicted in Fig. 1. The information bit stream is fed into a vector encoder and then followed by a beamforming module, which multiplies the encoded vector  $\mathbf{X}$  and the beamforming matrix  $\mathbf{P}$  to form the transmitting signal  $\mathbf{T} = \mathbf{P}\mathbf{X}$ . The channel  $\mathbf{H}$  is assumed to be the Rayleigh flat fading channel, i.e., the entries of  $\mathbf{H}$  are independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian variables with zero mean and unit variance ( $\mathcal{CN}(0, 1)$ ) and  $\mathbf{H}$  is i.i.d. for each channel use<sup>1</sup>. Let  $\mathbf{Y} \in \mathbb{C}^{L_R \times 1}$  be the received signal and  $\mathbf{W} \in \mathbb{C}^{L_R \times 1}$  be the Gaussian noise, then

$$\mathbf{Y} = \mathbf{H}\mathbf{P}\mathbf{X} + \mathbf{W},$$

where  $E[\mathbf{W}\mathbf{W}^\dagger] = \mathbf{I}_{L_R}$ .

We also assume that at the beginning of each channel use, the channel state  $\mathbf{H}$  is perfectly estimated at the receiver, quantized to finite bits and then fed back to the transmitter through a feedback channel. The feedback is causal, error-free and zero-delay with rate up to  $R_{\text{fb}}$  bits/channel use. According to the channel state feedback, the transmitter decides the dimension of the encoded vector  $\mathbf{X}$ , say  $s$ , and chooses an appropriate beamforming matrix  $\mathbf{P}$  from a beamforming codebook  $\mathcal{B} = \cup_{s=0}^{L_T} \{\mathbf{P}_i \in \mathbb{C}^{L_T \times s} : \mathbf{P}_i^\dagger \mathbf{P}_i = \mathbf{I}_s\}$  where  $|\mathcal{B}| \leq 2^{R_{\text{fb}}}$  to satisfy the finite feedback constraint. Since the dimension of the encoded vector  $\mathbf{X}$  must match the rank of the beamforming matrix  $\mathbf{P}$ , i.e.,  $s = \text{rank}(\mathbf{P})$ , it is sufficient for the receiver to feedback the index of beamforming matrix. Define the feedback function as  $\varphi : \{\mathbf{H}\} \rightarrow \{i : 1 \leq i \leq 2^{R_{\text{fb}}}\}$ . This paper is going to quantify the information rate,

$$\max_{P_{\text{on}}} \max_{\mathcal{B}} \max_{\varphi(\cdot)} E_{\mathbf{H}} \left[ \ln \left| \mathbf{I}_{L_R} + P_{\text{on}} \mathbf{H} \mathbf{P}_{\varphi(\mathbf{H})} \mathbf{P}_{\varphi(\mathbf{H})}^\dagger \mathbf{H}^\dagger \right| \right],$$

subjecting to the average power constraint,

$$E_{\mathbf{H}} \left[ P_{\text{on}} \text{trace} \left( \mathbf{P}_{\varphi(\mathbf{H})} \mathbf{P}_{\varphi(\mathbf{H})}^\dagger \right) \right] = P_{\text{on}} E_{\mathbf{H}} [s] \leq \rho.$$

Since the variance of the Gaussian noise is normalized to 1, the average power constraint  $\rho$  is also the average received SNR.

<sup>1</sup>This i.i.d.  $\mathbf{H}$  assumption is for the block fading channel in practice when the channel state can be estimated and feedback for every fading block prior to payload transmission.

## III. POWER ON/OFF STRATEGY WITH PERFECT BEAMFORMING

This section considers power on/off strategy with perfect beamforming. The overall goal of this paper is to discuss the power on/off strategy with finite rate feedback, which implies imperfect beamforming. To isolate the effect of power on/off from the effect of imperfect beamforming, this section considers the perfect beamforming case.

Perfect beamforming is described as follows. The channel state matrix  $\mathbf{H}$  can be decomposed as  $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^\dagger$  where  $\mathbf{U} \in \mathbb{C}^{L_R \times L_R}$  and  $\mathbf{V} \in \mathbb{C}^{L_T \times L_T}$  are the unitary singular-vector matrices and  $\mathbf{\Lambda} \in \mathbb{C}^{L_R \times L_T}$  is the nonnegative diagonal singular-value matrix. Perfect beamforming means that the  $s$  columns of the beamforming matrix  $\mathbf{P} \in \mathbb{C}^{L_T \times s}$  are drawn from the right singular-vector matrix  $\mathbf{V}$ , i.e.  $\mathbf{V}^\dagger \mathbf{P} \in \mathbb{C}^{L_T \times s}$  with entries either 1 or 0.

In the perfect beamforming case, the optimal feedback function can be derived. For the purpose of normalization, define  $m \triangleq \min(L_R, L_T)$ ,  $\bar{s} \triangleq \frac{s}{m}$  and  $\mathbf{W} \triangleq \frac{1}{m} \mathbf{H}\mathbf{H}^\dagger$  if  $L_R < L_T$  or  $\mathbf{W} \triangleq \frac{1}{m} \mathbf{H}^\dagger \mathbf{H}$  if  $L_R \geq L_T$ . Denote the  $i^{\text{th}}$  largest eigenvalue of  $\mathbf{W}$  as  $\lambda_i$ . When beamforming is perfect, the MIMO channel  $\mathbf{H}$  can be viewed as a set of parallel sub-channels. The feedback only needs to indicate which sub-channels should be turned on. Since we have assumed that the power on each sub-channel is fixed to  $P_{\text{on}}$ , it is always better to turn on the beams corresponding to larger  $\lambda_i$ 's. Therefore, for a given  $\bar{P}_{\text{on}}$ , the optimal  $\bar{s}$  function is of the form

$$\bar{s} = \frac{1}{m} |\{k : \lambda_k \geq \kappa\}|, \quad (1)$$

where  $\kappa$  is the appropriate threshold chosen to satisfy the average power constraint

$$\bar{P}_{\text{on}} E_{\mathbf{H}} [\bar{s}] = \rho.$$

The corresponding information rate is

$$\mathcal{I} = E_{\mathbf{H}} \left[ \sum_{i=1}^s \ln(1 + \bar{P}_{\text{on}} \lambda_i) \right]. \quad (2)$$

Although the form of the optimal  $\bar{s}$  function is given in (1), it is difficult to evaluate the key parameters  $\kappa$  and  $\bar{P}_{\text{on}}$  and the corresponding performance  $\mathcal{I}$ . The calculation is usually done by Monte Carlo simulation. To overcome this difficulty, we let the numbers of transmit and receive antennas approach infinity simultaneously and do asymptotic analysis.

The following theorem shows that the optimal  $\bar{s}$  function converges to a constant, which is independent of the specific channel realization  $\mathbf{H}$ , for a given SNR when the numbers of transmit and receive antennas approach infinity simultaneously.

*Theorem 1:* Let  $m \triangleq \min(L_R, L_T)$ ,  $n \triangleq \max(L_R, L_T)$  and  $y \triangleq \frac{m}{n}$ . For a given SNR  $\rho$ , if  $m$  and  $n$  approach infinity simultaneously with  $y$  fixed, the optimal  $\bar{s}$  function converges to a constant,

$$\bar{s}_\infty \triangleq \lim_{(m,n) \rightarrow \infty} \bar{s} = \int_a^\pi f_T(t) dt$$

almost surely, and the corresponding normalized information rate  $\bar{\mathcal{I}} \triangleq \frac{1}{m} \mathcal{I}$  also converges to a constant,

$$\begin{aligned} \bar{\mathcal{I}}_\infty &\triangleq \lim_{(m,n) \rightarrow \infty} \bar{\mathcal{I}} \\ &= \int_a^\pi \ln \left( 1 + \frac{\rho}{y\bar{s}} (1 + y - 2\sqrt{y} \cos(t)) \right) f_T(t) dt, \end{aligned} \quad (3)$$

where

$$f_T(t) = \begin{cases} \frac{1}{\pi} \cdot \frac{1 - \cos(2t)}{1 + y - 2\sqrt{y} \cos(t)} & \text{if } y < 1 \\ \frac{1 + \cos(t)}{\pi} & \text{if } y = 1 \end{cases},$$

and the threshold  $a \in [0, \pi]$  is chosen to maximize (3).

The proof is not included in this paper due to the length limit. It can be found in the journal version of this paper [10].

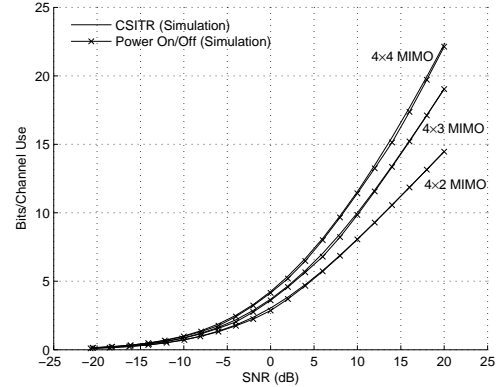
The key parameters and performance can be calculated by asymptotics. In [10], we derive asymptotic formulas to find the optimal threshold  $a_\infty$ , the optimal normalized number of on-beams  $\bar{s}_\infty$  and the corresponding normalized information rate  $\bar{\mathcal{I}}_\infty$ . Moreover, we prove that  $\bar{s}_\infty$  is a non-decreasing function of the average SNR  $\rho$  [10]. When  $\rho$  is small enough, at most one beam should be turned on; When  $\rho$  is sufficiently large, all beams should be on.

Theorem 1 can be applied to MIMO systems with finite many antennas and suggests a power on/off strategy with a constant number of on-beams. Similar to [11]–[13], our asymptotic results remain accurate when applied to finite systems. However, one should notice the difference between the asymptotic case and the finite case. The asymptotic  $\bar{s}_\infty$  can be any rational number in  $[0, 1]$  while  $\bar{s} \in \{\frac{1}{m}, \frac{2}{m}, \dots, 1\}$  in finite systems. The specific procedure to overcome this discrepancy is detailed in [10].

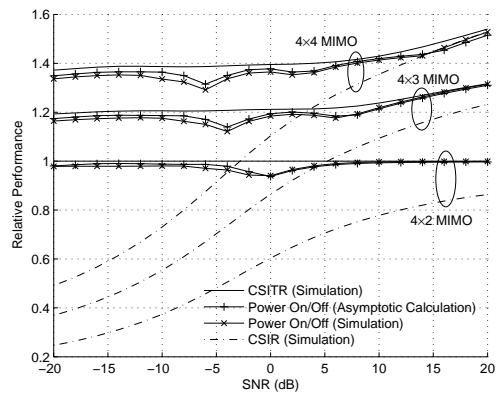
Simulations show that a constant number of on-beams is near optimal when beamforming is perfect. Fig. 2 compares the information rate of a constant number of on-beams and that of water filling power allocation (CSITR case) for  $4 \times 2$ ,  $4 \times 3$  and  $4 \times 4$  MIMO systems. The performance difference is almost unnoticeable according to the results in Fig. 2(a). To show the performance differences clearer, Fig. 2(b) presents the relative performance defined as the ratio of the considered information rate and the capacity of a  $4 \times 2$  MIMO achieved by water filling in all SNR regimes. It shows that a constant number of on-beams can achieve more than 90% of the capacity achieved by water filling in all SNR regimes. The maximum loss points are due to the fact that  $\bar{s}$  can only take discrete values. Furthermore, the performance approximated by asymptotic formulas is accurate.

#### IV. EFFECT OF FINITE SIZE BEAMFORMING CODEBOOK

This section considers the effect of imperfect beamforming. Perfect beamforming requires infinite feedback rate and therefore is unrealistic. In this section, finite size beamforming codebooks are considered. For power on/off strategy with a constant number of on-beams, the beamforming codebook  $\mathcal{B}$  contains matrices of the same rank. Describe such a beamforming codebook as a *single rank beamforming codebook*. To quantify its effect on information rate, the Grassmann manifold



(a) Average mutual information



(b) Relative performance

Fig. 2. Performance for power on/off strategy with fixed number of on-beams and perfect beamforming.

is introduced in IV-A and an asymptotic optimal feedback strategy is defined and discussed in Section IV-B.

#### A. Grassmann Manifold

The Grassmann manifold is the geometric object relevant to the beamforming codebook performance analysis. The Grassmann manifold  $\mathcal{G}_{L_T, s}(\mathbb{C})$  is the set of  $s$ -dimensional planes (passing through the origin) in complex Euclidean  $L_T$ -space  $\mathbb{C}^{L_T}$ . A generator matrix  $\mathbf{P} \in \mathbb{C}^{L_T \times s}$  for an  $s$ -plane  $\mathcal{P} \in \mathcal{G}_{L_T, s}(\mathbb{C})$  is the matrix whose columns are orthonormal and span  $\mathcal{P}$ . The generate matrix is not unique. That is, if  $\mathbf{P}$  generates  $\mathcal{P}$  then  $\mathbf{P}\mathbf{U}$  also generates  $\mathcal{P}$  for any  $s \times s$  unitary matrix  $\mathbf{U}$  [14]. The chordal distance between two  $s$ -planes  $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{G}_{L_T, s}(\mathbb{C})$  can be defined by their generator matrices  $\mathbf{P}_1$  and  $\mathbf{P}_2$  via  $d_c(\mathcal{P}_1, \mathcal{P}_2) = \frac{1}{\sqrt{2}} \left\| \mathbf{P}_1 \mathbf{P}_1^\dagger - \mathbf{P}_2 \mathbf{P}_2^\dagger \right\|_F$  [14]. The uniform distribution on  $\mathcal{G}_{L_T, s}(\mathbb{C})$  with density function  $f_{\mathcal{P}}(\cdot)$  satisfies  $f_{\mathcal{P}}(\mathcal{P}_1) = f_{\mathcal{P}}(\mathcal{P}_2)$  for arbitrary  $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{G}_{L_T, s}(\mathbb{C})$  [15].

#### B. Feedback Strategy

This subsection introduces a suboptimal but asymptotic optimal feedback strategy because the corresponding performance

can be well approximated. Consider the singular value decomposition that  $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^\dagger$ . Define  $\mathbf{V}_s$  as the  $L_T \times s$  matrix composed by the  $s$  vectors in  $\mathbf{V}$  corresponding to the largest  $s$  singular values. Then both  $\mathbf{V}_s$  and a beamforming matrix  $\mathbf{P} \in \mathcal{B}$  can be viewed as generator matrices of  $s$ -planes in  $\mathcal{G}_{L_T, s}(\mathbb{C})$ . Denote the planes generated by  $\mathbf{V}_s$  and  $\mathbf{P}$  as  $\mathcal{P}(\mathbf{V}_s)$  and  $\mathcal{P}(\mathbf{P})$  respectively. The feedback function is defined as

$$\hat{\varphi}(\mathbf{H}) \triangleq \arg \min_{1 \leq i \leq |\mathcal{B}|} d_c(\mathcal{P}(\mathbf{P}_i), \mathcal{P}(\mathbf{V}_s)). \quad (4)$$

Although this feedback strategy is not optimal in general, it is asymptotic optimal when the size of  $\mathcal{B}$  approaches infinity [10].

The following theorem defines a performance measure and gives a good approximation to that measure. Since  $\mathcal{P}(\mathbf{V}_s)$  is uniformly distributed in  $\mathcal{G}_{L_T, s}(\mathbb{C})$  [16], this theorem can be applied to the feedback function (4).

*Theorem 2:* Let  $\mathcal{B}$  be a single rank beamforming codebook with size  $K \triangleq |\mathcal{B}|$ . Define the average squared chordal distance as

$$\bar{d}_c^2(\mathcal{B}) \triangleq E_{\mathcal{V}} \left[ \min_{1 \leq i \leq |\mathcal{B}|} d_c^2(\mathcal{P}(\mathbf{P}_i), \mathcal{V}) \right],$$

where  $\mathcal{V}$  is uniformly distributed in  $\mathcal{G}_{L_T, s}(\mathbb{C})$ . Then the minimum average squared chordal distance achievable,

$$\bar{d}_{c, \text{inf}}^2 \triangleq \inf_{\mathcal{B}: |\mathcal{B}|=K} \bar{d}_c^2(\mathcal{B}),$$

can be bounded by

$$\frac{t}{t+1} \eta^{-\frac{1}{t}} 2^{-\frac{\log_2 K}{t}} \lesssim \bar{d}_{c, \text{inf}}^2 \lesssim \frac{\Gamma(\frac{1}{t})}{t} \eta^{-\frac{1}{t}} 2^{-\frac{\log_2 K}{t}}, \quad (5)$$

where  $t = s(L_T - s)$ ,

$$\eta = \begin{cases} \frac{1}{t!} \prod_{i=1}^s \frac{(L_T - i)!}{(s - i)!} & \text{if } 1 \leq s \leq \frac{L_T}{2} \\ \frac{1}{t!} \prod_{i=1}^{L_T - s} \frac{(L_T - i)!}{(L_T - s - i)!} & \text{if } \frac{L_T}{2} \leq s \leq L_T \end{cases},$$

and the symbol  $\lesssim$  denotes the *main order inequality*,  $f(K) \lesssim g(K)$  if  $\lim_{K \rightarrow +\infty} \frac{f(K)}{g(K)} \leq 1$ .

Although this theorem is for asymptotically large  $K$ , the bounds (5) are accurate enough for relatively small  $K$ . For example, it is shown in [17] that the bounds are tight for  $K \geq 10$  when  $L_T = 4$ ,  $s = 2$ . Furthermore, as the number of real dimensions of the Grassmann manifold ( $2t$ ) approaches infinity, both the lower bound and the upper bound converges to 1.

It is noteworthy that Theorem 2 holds for Grassmann manifolds with arbitrary dimensions. In [7], approximations to  $\bar{d}_{c, \text{inf}}^2$  are developed for  $s = 1$  case and the case that  $s \geq 1$  is fixed and  $L_T$  is asymptotically large. Indeed, the approximation in [7] for  $s = 1$  is a lower bound on  $\bar{d}_{c, \text{inf}}^2$ . The approximation in [7] for fixed  $s$  and asymptotically large  $L_T$  is neither a lower bound nor an upper bound. A detailed comparison of Theorem 2 and the results of [7] can be found in [17].

### C. Effect of a Beamforming Codebook

For a finite size single rank beamforming codebook, the following proposition quantifies its effect on information rate.

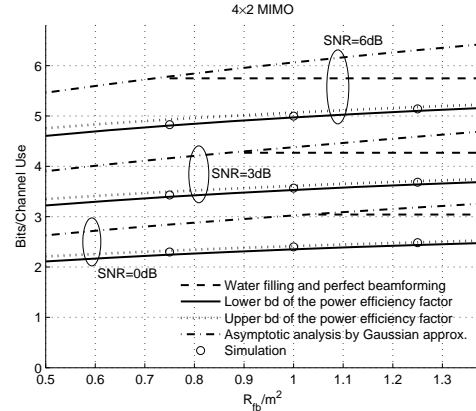


Fig. 3. Performance of finite size beamforming codebook.

*Proposition 1:* Consider the feedback strategy in (4). The normalized information rate of the power on/off strategy with a single rank beamforming codebook  $\mathcal{B}$  can be bounded by

$$\begin{aligned} \bar{\mathcal{I}} &= \frac{1}{m} E_{\mathbf{H}} \left[ \ln \left| \mathbf{I}_{L_R} + P_{\text{on}} \mathbf{H} \mathbf{P}_{\hat{\varphi}(\mathbf{H})} \mathbf{P}_{\hat{\varphi}(\mathbf{H})}^\dagger \mathbf{H}^\dagger \right| \right] \\ &\leq \frac{1}{m} E_{\mathbf{H}} \left[ \sum_{i=1}^s \ln(1 + \mu P_{\text{on}} \lambda_i) \right], \end{aligned} \quad (6)$$

where  $\lambda_i$  is the  $i^{\text{th}}$  largest eigenvalue of  $\frac{1}{m} \mathbf{H} \mathbf{H}^\dagger$  and the constant

$$\mu \triangleq 1 - \frac{1}{s} E_{\mathcal{V}} \left[ \min_{\mathbf{Q} \in \mathcal{B}} d_c^2(\mathcal{P}(\mathbf{Q}), \mathcal{V}) \right] \quad (7)$$

is called *power efficiency factor*, where plane  $\mathcal{V}$  is uniformly distributed in  $\mathcal{G}_{L_T, s}(\mathbb{C})$ .

The derivation can be found in [10]. The constant  $\mu$  is called power efficiency factor because the effect of a single rank beamforming codebook is to decrease the  $P_{\text{on}}$  in (2) to  $\mu P_{\text{on}}$  in (6). Applying Theorem 2, the maximum  $\mu$  achievable can be bounded as a function of the feedback rate  $R_{\text{fb}} = \log_2 K$ .

Combining the asymptotic formulas for perfect beamforming [10] and the bounds for power efficiency factor, the information rate of power on/off strategy with a constant number of on-beams can be accurately approximated. Fig. 3 compares the simulated information rate and the approximation for a  $4 \times 2$  MIMO system. We also compare our results to another performance approximation proposed in [9], which is based on Gaussian approximation and asymptotic analysis. Simulations show that our performance approximation almost perfectly matches the actual performance and is more accurate than the one proposed in [9].

## V. PERFORMANCE COMPARISON

While we have shown that power on/off strategy is near optimal under perfect beamforming in Section III, this section will show that it is also near optimal for MIMO systems with finite rate feedback.

We compare the power on/off strategies with single rank beamforming codebooks and multi-rank beamforming codebooks. A multi-rank beamforming codebook can be expressed

as  $\mathcal{B} = \bigcup_{s=0}^{L_T} \mathcal{B}_s$  where each sub-code  $\mathcal{B}_s$  is defined as  $\mathcal{B}_s \triangleq \{\mathbf{P}_i \in \mathbb{C}^{L_T \times s} : \mathbf{P}_i \in \mathcal{B}\}$ . Given a multi-rank beamforming codebook and  $P_{\text{on}}$ , we derive the optimal feedback strategy [10]<sup>2</sup>. However, the key parameters and the corresponding information rate need to be computed by Monte Carlo simulations. The optimal multi-rank beamforming codebook is also found by numerical search. Define  $K_s \triangleq |\mathcal{B}_s|$ . We try all possible combinations of  $K_s$ 's such that  $\sum_{s=0}^{L_T} K_s = 2^{R_{\text{fb}}}$ . For each possible codebook  $\mathcal{B} = \bigcup_{s=0}^{L_T} \mathcal{B}_s$ , we find the optimal key parameters and the corresponding information rate by simulation. Finally we choose the codebook corresponding to the best information rate.

Fig. 4 compares the information rate achieved by single-rank beamforming codebooks and multi-rank beamforming codebooks. The relative performance, defined as the ratio of the considered information rate and the capacity of a  $4 \times 2$  MIMO, is also given in Fig. 4(b). Simulations show that single rank beamforming codebooks are near optimal even when the feedback rate is finite. Therefore, the power on/off strategy with a constant number of on-beams is near-optimal for MIMO systems with finite rate channel state feedback.

## VI. CONCLUSIONS

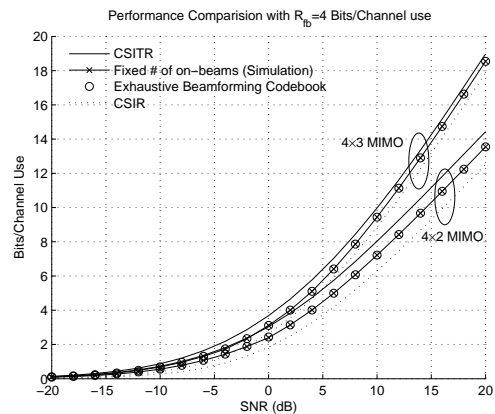
This paper shows that the power on/off strategy with a constant number of on-beams is near-optimal. By introducing power efficiency factor, this paper proposes a new way to quantify the effect of a single rank beamforming codebook. A good information rate approximation is presented in this paper.

An important point that is not mentioned in this paper is the complexity of selecting the feedback beamforming matrix in a codebook. To avoid exhaustive search, beamforming codebooks with certain structure may be considered in future.

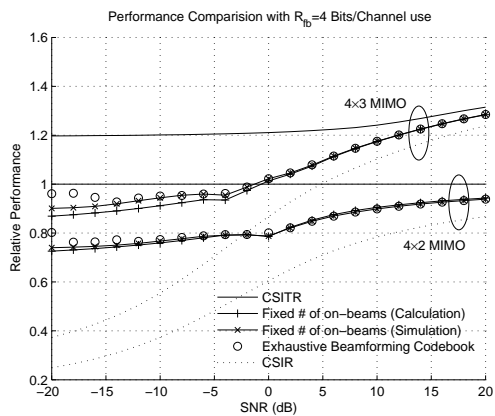
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<sup>2</sup>The difference between our approach and [5] is that we only put constraint on average power consumption while the transmitted power is fixed for every channel realization in [5].



(a) Average mutual information



(b) Relative performance

Fig. 4. Comparison of single rank beamforming codebooks and multi-rank beamforming codebooks.

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