3510 - Solutions to Sample HW Problems - Assignments 7 & 8

From Assignment 7:

10.3 If $X \sim Unif[0,1]$, $V = X/(1-X)$ takes values on $[0,\infty)$ and has distribution function:

$$P(X/(1-X) \le t) = P(X \le t/(1+t)) = \frac{t}{1+t}.$$

So, the density function for V is $f_V(t) = \frac{d}{dt}\frac{t}{1+t} = \frac{1}{(1+t)^2}$ for $t \in [0,\infty)$ and otherwise is equal to zero.

Set now $W = X(1-X)$. This r.v. takes values between 0 and 1/4. So, with $t \in [0,1/4]$ we compute

$$P(W \le t) = P(X^2 - X + t \ge 0) = 1 - P\left(\frac{1-\sqrt{1-4t}}{2} \le X \le \frac{1+\sqrt{1-4t}}{2}\right) = 1 - \sqrt{1-4t}.$$

Hence, the density function for W is given by

$$f_W(t) = \frac{4}{\sqrt{1-4t}} \text{ if } t \in [0,1/4], \quad f_W(t) = 0 \text{ otherwise.}$$

10.5 Here, U_1 and U_2 are independent $Unif[0,1]$ r.v.'s. First note,

$$P(\max(U_1,U_2) \le t) = P(U_1 \le t \cap U_2 \le t) = P(U_1 \le t)^2 = t^2,$$

thus the density of the maximum is $2t$ for $t \in [0,1]$. Similarly,

$$P(\min(U_1,U_2) \le t) = 1 - P(\min(U_1,U_2) > t) = 1 - P(U_1 > t)^2 = 1 - (1-t)^2,$$

and the density of the minimum is $2(1-t)$, again on $[0,1]$.

10.10 Let us first compute $P(V \le t)$. This is the proportion of the area of the circle $x^2 + y^2 \le 1$ given by $-t \le x \le t$. So, we could compute 2 times the area under the curve $y = \sqrt{1-x^2}$ between $-t$ and t (then divide by π). That is,

$$P(V \le t) = \frac{2}{\pi}\int_{-t}^{t} \sqrt{1-x^2}dx.$$

Taking derivatives, we find the density of V is $f_V(t) = \frac{4}{\pi}\sqrt{1-t^2}$. Thus

$$E[V] = \int_0^1 t\frac{4}{\pi}\sqrt{1-t^2}dt = \frac{2}{\pi}.$$

10.14 The information given is equivalent to being told the density function of X is

$$f_X(x) = \frac{1}{4}(x-5)^2 + \frac{1}{6} \text{ for } 5 \le x \le 7,$$

and otherwise is 0. Therefore

$$E[X] = \int_5^7 x\left[\frac{1}{4}(x-5)^2 + \frac{1}{6}\right]dx = \frac{19}{3}.$$

Then

$$Var[X] = \int_0^7 x^2\left[\frac{1}{4}(x-5)^2 + \frac{1}{6}\right]dx - \left(\frac{19}{3}\right)^2,$$

but I won't work it out.

10.20 Let $U \sim Unif[0,1]$ and $V = \sqrt{U}$, $W = U^2$. We have

$$E[V] = \int_0^1 \sqrt{x}dx = 2/3, \quad Stdv[V] = \sqrt{\int_0^1 xdx - 4/9} = 1/\sqrt{18}.$$

and

$$E[W] = \int_0^1 x^2dx = \frac{1}{3}, \quad Stdv[W] = \sqrt{\int_0^1 x^4dx - 1/9} = 2/(3\sqrt{5}).$$

From Assignment 8

9.26 You must sum three independent geometric random variables, each with parameter $p = 1/6$. The result is what is called a negative Binomial random variable. It takes values $k = 3, 4, 5, \ldots$ and has mass function

$$p(k) = \binom{k-1}{2}(5/6)^{(k-3)}(1/6)^3.$$

(The kth roll must be a 6, of the remaining $k-1$ you must choose 2 places to put 6's. Then overall there are 3 6's which occur with prob 1/6, and $k-3$ non-sixes which occur with prob 5/6.)

9.29 The exact probability is given by

$$1 - \frac{\binom{124900}{2500}}{\binom{125000}{2500}},$$

but you can also use a Poisson approximation. You have 100 chances, with the probability of any being a success equal to $2500/125000 = 1/50$. So the number of winning tickets you have can be modeled as a $Poisson(2)$. Thus,

$$P(\text{win}) = 1 - P(\text{no winning ticket}) \approx 1 - e^{-2}.$$

9.30 You are blindly choosing 10 of 15 problems to study, your hope is that at least 5 of these come from the 8 "good" problems. Hence your chance of passing is

$$P(\text{choose 5, 6, 7, or 8 good problems}) = \sum_{k=5}^{8} \frac{\binom{8}{k}\binom{7}{10-k}}{\binom{15}{10}}.$$

4.20 Any two of the 625 drawings have a 1/1000 chance of being the same. There are $\binom{625}{2} = 625 \cdot 312$ pairs of drawing.

Thus, the number of pairs of drawings that are the same is approximately Poisson with parameter $\frac{625 \cdot 312}{10000} = 19.5$. And the prob that no two drawings are the same is $1 - e^{-19.5}$.

4.21 In 25 people, there are $25 \cdot 23 \cdot 4 = 2300$ groups of three. The probability that three people share a birthday is $(365)^{-2}$. So we are looking at a Poisson $2300/(365^2) = .0173$. The probability we are after is them $1 - e^{-(.0173)}$.

Now if instead we are interested in the three people of being born within one day of each other....the probability of this "success" is $7 \cdot (365)^{-2}$ since, fixing one person's birthday there are 7 different ways to give two other people birthdays within one day before or after that chosen day. So now must use a $Poisson(.121)$, and the desired prob is $1 - e^{-.121}$.