3510 - Solutions to Sample HW Problems - Assignments 5 & 6

From Assignment 5

8.16 We are being asked

$$\begin{aligned} P(\text{rain} \mid \text{rain predicted}) &= \frac{P(\text{rain predicted}|\text{ rain})P(\text{rain})}{P(\text{rain predicted})} \\ &= \frac{P(\text{rain predicted} \mid \text{rain})P(\text{rain})}{P(\text{rain predicted} \mid \text{rain})P(\text{rain}) + P(\text{rain predicted}|\text{ no rain})P(\text{no rain})} \\ &= \frac{(.85)(.1)}{(.85)(.1) + (.25)(.9)}. \end{aligned}$$

8.17 We wish to compute

$$\begin{aligned} P(\text{not fair} \mid \text{saw HHH}) &= \frac{P(\text{saw HHH} \mid \text{not fair})P(\text{not fair})}{P(\text{saw HHH})} \\ &= \frac{P(\text{saw HHH} \mid \text{not fair})P(\text{not fair})}{P(\text{saw HHH} \mid \text{not fair})P(\text{not fair}) + P(\text{saw HHH} \mid \text{fair})P(\text{fair})} \\ &= \frac{(3/4)^3(1/5)}{(3/4)^3(1/5) + (1/2)^3(4/5)}. \end{aligned}$$

6.5 Have Monty Hall with 5 doors where Monty opens 2 doors after your initial choice. The probability of it being behind one of the four doors you don't choose initially is .8. So, after he opens two doors, there is .8 chance that the prize is in one of the other two remaining (the two excluding your initial choice and the ones he opened). So by switching you win with probability .4 (as the two remaining doors share equally in the .8 chance).

6.6 Here there are four doors and he'll open two in succession, allowing you the chance to switch each time. The optimal strategy is to first stay put, then switch after he opens the second door. The reasoning is the same. There is a 3/4 chance the prize was in one of the three doors you didn't pick at the beginning. By forcing him to open two of these, you are "exposing" which of these doors gets all the mass 3/4.

From Assignment 6

9.9 Indicator trick:

$$\begin{aligned} E[\#\text{distinct B-days}] &= E\left[\sum_{k=1}^{365} \mathbf{1}_{\{\text{someone born on day k}\}}\right] \\ &= 365 \cdot P(\text{at least one of 100 people born on day \# 1}) \\ &= 365 \cdot (1 - P(\text{ none of 100 people born on day \# 1 }) \\ &= 365\Big(1 - \Big(\frac{364}{365}\Big)^{100}\Big). \end{aligned}$$

9.10 Use indicator trick again:

$$E[\text{number of pairs in row sharing same letter}] = 10 \cdot P(\text{letter 1 and letter 2 are same}),$$

and

$$\begin{aligned} P(\text{letter 1 and letter 2 are same}) &= P(M, \text{then } M) + P(I, \text{then } I) + P(P, \text{then } P) \\ &= \frac{4 \cdot 3}{11 \cdot 10} + \frac{4 \cdot 3}{11 \cdot 10} + \frac{2 \cdot 1}{11 \cdot 10} \\ &= 26/110. \end{aligned}$$

So the expectation is 260/110.

9.13 The number of trials in this example is a $Geom(3/4)$ random variable. The variance of a $Geom(p)$ is $(1-p)/p^2$. Hence the standard deviation in this instance is $\sqrt{(1/4)/(9/16)} = 2/3$.

10.2 We want to find the distribution of both $X + Y$ and XY when X and Y are each independent and uniform on $[0, 1]$.

The first question amounts to what proportion of the square $[0, 1] \times [0, 1]$ is given by $x + y \leq t$:

$$P(X + Y \leq t) = \text{ area under the line } y = t - x \text{ for which } x, y \in [0, 1] \times [0, 1].$$

This equals t^2 if $t \in [0, 1]$ and equals $(1 - t)^2$ for $t \in [1, 2]$. Therefore the probability density function is $f(t) = 2t$ for $t \in [0, 1]$, $f(t) = 2(1 - t)$ for $t \in [1, 2]$, $f(t) = 0$ for $t < 0$ or $t > 2$.

The second question is similar: what portion of the square is described by $xy \leq t$. This can be thought of as the area under the curve $y = t/x$, but bounded by the square. Noting in this case t can only be between 0 and 1:

$$P(XY \leq t) = t + \int_t^1 (t/x)dx = t - t \ln t.$$

Hence, the probability density function of XY is $-\ln t$ for $t \in [0, 1]$ and zero otherwise.