3510 - Solutions to Sample HW Problems - Installment 2

September 26, 2009

From Assignment 3:

7.6 We can take as the sample space the 60×60 square in the $x-y$ plane. (We are working in minutes, the x coordinate is how long past 12 : 00 the first person shows up, the y coordinate the same for person 2.) The event in question − that the two show up within 10 minutes of each other − corresponds geometrically to the strip $x - 10 \leq y \leq x + 10$. The probability being the ratio of the areas we have

$$P(\text{x and y are within 10 minutes of each other}) = \frac{3600 - 2500}{3600} = \frac{11}{36}.$$

7.7 The condition for a real root is that $B^2 \geq 4C$. In the first part B and C are each drawn from $[-1, 1]$, so in this case the sample space has area 4 while the area of the portion of that space defined by $B^2 \leq 4C$ is given by

$$2 + \frac{1}{4}\int_{-1}^{1} B^2 dB = 2 + 1/6 = 13/6.$$

Hence, the probability is 13/24.

More generally, if B, C are drawn from $-q, q$ we that, if $q \leq 4$ the probability is

$$\frac{2q^2 + \frac{1}{4}\int_{-q}^{q} B^2 dB}{4q^2} = \frac{1}{2} + \frac{q}{24}.$$

If $q > 4$ a picture shows we should instead compute

$$\frac{2q^2 + 2q(q - 2\sqrt{q}) + \frac{1}{4}\int_{-2\sqrt{q}}^{2\sqrt{q}} B^2 dB}{4q^2} = 1 - \frac{2}{3}\frac{1}{\sqrt{q}}.$$

Note that the difference is that when $q > 4$, the parabola $C = B^2/4$ leaves the $[-q, q] \times [-q, q]$ box out the top, rather than through the sides.

7.8 The sample space is the entire $20cm \times 50cm$ board and the event in question is whether we fall in any of the quarter circles of radius $5cm$ in any corner. So, by computing the proportion of the areas, the probability is

$$\frac{25 \cdot \pi}{1000} = \frac{\pi}{40}.$$

7.9 (I'll just do the first part; the second part is more an exercise in geometry than in probability.)

If the perpendicular distance of the random point to the base is larger than d, then the random point must be drawn from the similar triangle (with the same "peak") of height $h-d$ and base length $b \cdot (\frac{h-d}{d})$. So the probability is the ratio of the area of this triangle to the original.

$$\frac{(h-d) \cdot (b \cdot \frac{h-d}{d})}{h \cdot b} = \frac{(h-d)^2}{h^2}.$$

7.12 Let's use the notation $A > B$ for "A beats B" etc. We wish to compute

$$\begin{aligned} &P(A \text{ wins as many games as } B \text{ or } C) \\ &= P(A > B > C > A) + P(A > C > B > A) + P(A > B \cap A > C), \end{aligned}$$

as the event that A wins at least as many games as the other two is the same as A wins two games or everyone wins exactly one game. By independence the above is equal to

$$\begin{aligned} &P(A > B)P(B > C)P(C > A) + P(A > C)P(C > B)P(B > A) + P(A > B)P(A > C) \\ &= (.5)(.4)(.3) + (.7)(.6)(.5) + (.5)(.7) = .62 \end{aligned}$$

7.14 I'll do the first two parts. First

$$\begin{aligned} P(\text{see 6 before 7}) &= \sum_{k=1}^{\infty} P(\text{see 6 on roll k, no 6 or 7 before}) \\ &= \sum_{k=1}^{\infty} P(\text{no 6 or 7})^{k-1} P(\text{see 6}) \\ &= \sum_{k=1}^{\infty} (25/36)^{k-1}(5/36) = 5/11. \end{aligned}$$

Next, when they write "what about an 8 and a seven" I take it to mean that we get a six before ever seeing *both* an 8 and a 7 (it is OK to see some 7's or 8's before the 6, but not OK to see some 7's *and* some 8's). We then need

$$P(\text{see 6 before 7}) + P(\text{see 6 before 8}) - P(\text{see 6 before 8 or 7}),$$

as the first two count not seeing any 7's first, then not seeing and 8's first also both include that we not see any 7's or 8's.

We've already computed the first of these probabilities, and the other two are similar. Since $P(\text{see no 6 or 8}) = 26/36$, the second probability above is $5/10$ and since $P(\text{see no } 6, 7, \text{ or } 8) = 20/36$ the last probability is $5/16$. Thus, the final answer is $5/11 + 5/10 - 5/16$.

7.15 We are given $P(A \cap B) = .3$ and $P(A) = .7$, $P(B) = .5$. So,

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) = .1.$$

7.16 If $P(A) = 2/3$ and $P(A \cup B) = 3/4$, there are two extremes: either A is contained in B or A and B are disjoint. In the former case, $P(B) = 3/4$, and in the latter case $P(B) = 3/4 - 2/3 = 1/12$. These are the largest and smallest possible values for the probability of B.

From Assignment 4:

7.17 Write what is given as $P(T) = .75$, $P(T \cap V) = .3$, and $P(T^c \cap V^c) = .1$. We then want $P(V)$. But

$$P(V) = P(V \cap T) + P(V \cap T^c) = P(V \cap T) + (1 - P(T) - P(T^c \cap V)) = .3 + .15 = .45.$$

(Might help to draw a picture here.)

7.19 We write $P(3)$ as shorthand for the probability the number is divisible by 3 and so on. We have

$$P(3 \cup 5) = P(3) + P(5) - P(3 \cap 5) = \frac{333}{1000} + \frac{200}{1000} - \frac{66}{1000} = \frac{467}{1000}.$$

If you want to compute the probability that you are divisible by $3, 5$, or 7 must you inclusion-exclusion with three events:

8.3 The first part asks

$$P(\text{only one Ace } | \text{at least one Ace}) = \frac{P(\text{only one Ace})}{P(\text{at least one Ace})}.$$

Plainly,

$$P(\text{only one Ace}) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}$$

and

$$P(\text{at least one Ace}) = 1 - P(\text{no Ace}) = 1 - \frac{\binom{48}{13}}{\binom{52}{13}}$$

which can be substituted into the above.

On the other hand,

$$P(\text{only one Ace }|\text{have Ace of Hearts}) = \frac{P(\text{only one Ace, and is Hearts})}{P(\text{have Ace of Hearts})} = \frac{\binom{48}{12}}{\binom{51}{12}}.$$

If you compute things out, you'll that these conditional probabilities yield different numbers (it should strike you as interesting that knowing that there is an Ace vs. knowing that there is a specific kind of Ace changes the probabilities).

8.4 By the basic rule

$$P(\text{lost at Dubai} \mid \text{lost}) = \frac{P(\text{lost at Dubai})}{P(\text{lost})} = \frac{(.95)(.3)}{.05 + (.95)(.3) + (.95)(.97)(.02)}.$$

Note here that in computing the probability it was lost at a specific place you must take into account it was not lost earlier.

8.6 The probability that your team doesn't make it is

$$(1 - p_1) + p_1(1 - p_2) + p_1 \cdot p_2(1 - p_3) + p_1 \cdot p_2 \cdot p_3(1 - p_4).$$

So, knowing that they didn't make it, the probability they were knocked out in round two (for example) is

$$\frac{p_1(1 - p_2)}{(1 - p_1) + p_1(1 - p_2) + p_1 \cdot p_2(1 - p_3) + p_1 \cdot p_2 \cdot p_3(1 - p_4)}.$$

The probability that they are knocked out at different rounds is similar.

8.7 These events are independent because the coins are fair. Just knowing whether the first coin is and H or T has no influence on what the second coin produces, and so (by fairness), doesn't impact the probability that the second coin will take the same value as the first.

8.8 Decompose the event in terms of the value of the die:

$$\begin{aligned} P(\text{see no H's}) &= \sum_{k=1}^{6} P(\text{see no H's} \mid \text{roll 6}) P(\text{roll 6}) \\ &= (1/6) \sum_{k=1}^{6} P(\text{see no H's in k tries}) \\ &= (1/6) \sum_{k=1}^{6} (1/2)^k = \frac{21}{128}. \end{aligned}$$