

3510 - Solutions to Sample HW Problems - Installment 1

September 20, 2009

From Assignment 1:

2.2 Every contestant being equally likely to make the top four:

$$P(\text{Sheila makes top four}) = \frac{\# \text{groups of 4 which include Sheila}}{\# \text{total groups of 4 from 12 people}} = \frac{\binom{11}{3}}{\binom{12}{4}} = 1/3.$$

Alternatively $P(\text{Sheila picked first}) = 1/12$, and $P(\text{Sheila picked second}) = (11 \cdot 1)/(12 \cdot 11) = 1/12$, with similar reasoning showing that the probability she is picked third or fourth are also both $1/12$.

2.3 The sample space here is four-letter long strings, each letter being and “M” or “F”, for a total of $2^4 = 16$ elements. (Puppies are distinct things, so we take order to matter.) So

$$P(2 \text{ “M” ’s and } 2 \text{ “F” ’s}) = \frac{\binom{4}{2}}{2^4} = 6/16 = 3/8,$$

while

$$P(3 \text{ “M” ’s and } 1 \text{ “F”}) = \frac{\binom{4}{1}}{2^4} = 4/16 = 1/4,$$

with the probability of 3 “F”’s and 1 “M” the same. Thus, the probability of seeing 3 of one gender and 1 of another is $1/2$ which is bigger than the $3/8$ probability of seeing two of each.

2.4 (a) Each of three people gets two choices (Plumber A or Plumber B) for a total of $2^3 = 8$ elements in the sample space. There are exactly two ways they can all make the same choice: each picks A or each picks B. So, the probability they all pick the same plumber is $2/8 = 1/4$.

(b) We call R_1, R_2, R_3 the values of the first, second, and third rolls respectively. We wish to compute $P(R_1 < R_2 < R_3)$. We index the possible ways the event of interest can happen via R_2 :

If $R_2 = 2$, R_1 must be 1, and R_3 could be 3, 4, 5, 6.

If $R_2 = 3$, R_1 could be 1, 2 and R_3 could be 4, 5, 6.

If $R_2 = 4$, R_1 could be 1, 2, 3 and R_3 could be 5, 6.

If $R_2 = 5$, R_1 could be 1, 2, 3, 4 and R_3 must be 6.

We have enumerated 20 elements in the event. Clearly the sample space has $6^3 = 216$ elements, and the probability is $20/216$.

(c) We need to compute the probability that A wins (and see whether this is $1/2$). Now

$$P(A\text{wins}) = P(\text{dif} = 0) + P(\text{dif} = 1) + P(\text{dif} = 2)$$

where “dif” is the numerical difference between the two rolls. There are 6 ways for dif= 0, 10 ways for dif= 1 and 8 ways for dif= 2. So

$$P(A\text{wins}) = \frac{6 + 10 + 8}{36} = 24/36.$$

2.5 It makes no difference whether you pick first or last. If you pick first, you may consider your sample space consisting of just 10 options, with just one of them being the winner so $P(\text{win, pick first}) = 1/10$. On the other hand if you go last, the sample space now consists of the $10!$ possible orderings for everyone, including you, to pick. And now, the number of ways that people can pick so that you win is $9!$ as the 9 people that go before you must choose from the 9 non-winning tickets in any order. Hence $P(\text{win, pick last}) = 9!/10! = 1/10$.

2.6 In ‘exact order’ you win with probability $1/(10 \cdot 9 \cdot 8) = 1/720$. In ‘any order’ win win with probability $3!/10^3 = 1/120$ (we multiplied by the number of re-orderings of three distinct objects).

So, under ‘exact order’

$$E[\text{win}] = \$360 \cdot (1/720) - \$1 \cdot (719/720) = -\$(361/720).$$

And under ‘any order’

$$E[\text{win}] = \$160 \cdot (1/120) - \$1 \cdot (119/120) = \$(41/120).$$

3.4 Let’s calculate the probability that you never win: by independence,

$$P(\text{don't win time 1} \cap \text{don't win time 2} \cap \text{don't win time 3} \cap \text{don't win time 4}) = (6/10)^4.$$

This is about 13%; seems to me you don’t have an unreasonable chance of not winning.

3.5 I’m going to allow numbers to start with zero. Given that, there are 10^5 different 5-digit numbers. If the first number is to equal and the second to equal the second last, it is as if

you are picking a three digit number (the last two being determined), and there are 10^3 such numbers. Thus the probability asked for is $10^{-2} = 1/100$.

Problems from Assignment 2:

2.7 The probability of rolling a sum of 2, 3, 4, 5, or 6 is the same as the probability of rolling a 8, 9, 10, 11, or 12. Namely, both probabilities are $15/36$. Further the chance of rolling a 7 is $6/36$. Therefore, for either bet

$$E[\text{win}] = \$2 \cdot (15/36) + \$0 \cdot (21/36) = \$(30/36).$$

2.11 The constraints being that you don't tell the rumor to yourself or the person who told you, the first person has 24 options and each of the following people have 23 options of who to tell. Thus the sample space in this model consists of $24 \cdot 23^9$ elements (ordered lists of the people who hear the rumor).

For no one to hear it twice, the first person can still tell 24 people, the second person 23, but now the third person only has 22 possible people to tell and so on. Thus,

$$P(\text{there are no repeats}) = \frac{(24 \cdot 23 \cdot 22 \cdot 21 \cdots 15)}{(24 \cdot 23^9)} = \prod_{k=1}^8 (1 - k/23).$$

If instead you just want the chance that person one never gets told the rumor: person one still has 24 options, person two has 23, and thereafter everyone who hears the rumor has 22 possibilities:

$$P(\text{never gets back to originator}) = (22/23)^8.$$

3.9 This is a poorly worded problem: you want to see two cars (car 3 and car 12 say) which both have, for example, first digit 2 and last digit 7. That is, you are not looking for the chance that one car has both the first and last digit equal to 2 and some other car has first and last digit equal to 7.

That said, this is a birthday problem with 15 "people" and $10 \cdot 10 = 100$ "days". So

$$P(\text{at least two matches}) = 1 - P(\text{no matches}) = 1 - \frac{100 \cdot 99 \cdot 98 \cdots 86}{100^{15}}.$$

3.10 There are 37 numbers on the roulette wheel. To ask for some number to come up at least twice in 10 spins is asking the birthday problem with 10 "people" and 37 "days". Thus, the probability asked for is

$$1 - \frac{37 \cdot 36 \cdots 28}{37^{10}}.$$

3.11 (a) Yet another birthday problem. There are seven actual people and now they are labeled by one of 25 possible floors. So

$$P(\text{at least 2 on the same floor}) = 1 - \frac{25 \cdot 24 \cdots 19}{25^7}.$$

(b) We still compute through the opposite:

$$P(\text{someone has the same floor as you}) = 1 - P(\text{no one has the same floor as you}) = 1 - (24/25)^6.$$

3.14 (a) So there are 10^4 possible winning numbers and 10 drawings. Another birthday problem with 10 "people" and 10000 days. So,

$$P(\text{see same winners somewhere}) = 1 - \frac{10000 \cdot 9999 \cdots 9991}{10000^{10}}.$$

(b) Let's call the probability computed in the first part ρ (I don't care what it is). We repeat the experiment of the first part 300 times (independently) and ask for the probability that the result (seeing the same number in two different states) occurs at least twice. Again, it is easier to compute the probability that it never happens, and subtract this from 1. That is, the desired probability is

$$1 - (1 - \rho)^{300}.$$