Math 2001, Spring 2009 **Sample Final**

1. Prove by induction: for any $n \geq 0$,

$$n^3 + (n+1)^3 + (n+2)^3$$

is divisible by 9.

2. Prove that for $x_1, x_2, \ldots, x_n$ integers, if $a = \frac{x_1 + \cdots + x_n}{n}$ there must be at least one k for which $x_k \geq a$.

3. Given the sequence

$$a_n = 3a_{n-1} - 2a_{n-2}$$

with $a_1 = 3, a_2 = 7$, prove that $a_n = 2^{n+1} - 1$.

4. Use Euclid's algorithm to find the gcd of 103 and 691. Also write $gcd(103, 691)$ in the form $A \cdot 103 + B \cdot 691$ for integers A and B.

5. How many ways can 6 men and 6 women be seated around a round table so that gender alternates as you look around the table?

6. How many straights are there in poker? In a straight you have 5 cards in a sequence (they need not be in the same suit); the ace can be high or low, but not in the middle of the sequence. What is the probability of getting a straight?

7. Prove the smallest cycle in a hypercube of dimension n has length 4.

8. Prove there can exist no bijection between the elements of a finite set A and that of its power set $P(A)$.

9. Is $K_{2,3,2}$ planar? How about $K_{2,3,2,3}$?

10. A graph G is such that between any two nodes there is exactly one path. Prove G is a tree.

11. Solve the system $X \equiv 4 (\mathrm{mod}\, 7), X \equiv 6 (\mathrm{mod}\, 11), X \equiv 2 (\mathrm{mod}\, 13)$.

12. Prove: For the full binary tree the number of leaves equals the number of internal nodes (that is, nodes that are not leaves) plus one.

13. Express the principle of Modus Ponens in the propositional calculus and show it is a tautology.

14. Know your definitions: function, inverse function, bijection, injection, surjection, relation, partial order, equivalence relation; subset, union, intersection, complement, power set; connectedness, cycle, node coloring, Eulerian cycle, tree, complete graph, etc, etc.