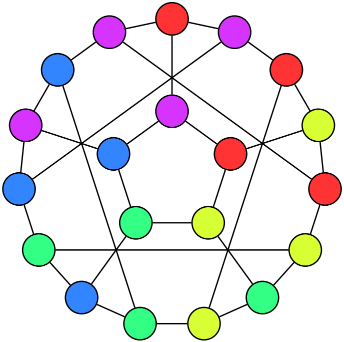
Topology in promise CSPs

Andrei Krokhin, Jakub Opršal, Marcin Wrochna, Standa Živný

Durham University

University of Oxford, UK

Graph coloring



Map vertices $V(G)$ to colors $\{1, 2, 3, 4, 5\}$
so that adjacent vertices get different colors.

Promise graph coloring

(Search version)

Given a 3-colorable graph G, find a 100-coloring.

(Decision version)

Distinguish 3-colorable graphs
from those that are not even 100-colorable.

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$3 \text{ vs } 5$ is NP-hard	Barto, Bulín, Krokhin, Opršal '18
$c \text{ vs } c'$ is NP-hard for all constants $3 \leq c \leq c'$?	*conjecture*
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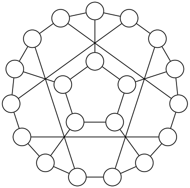
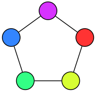
Motivation: constraint satisfaction, hardness of apx., codes, ...

Graph homomorphism $f\colon G \to H$

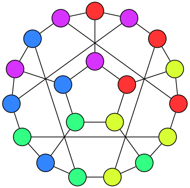
a function $f\colon V(G) \to V(H)$
such that $uv \in E(G) \implies f(u)f(v) \in E(H)$



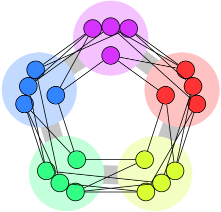
G



C_5



$G \to C_5$



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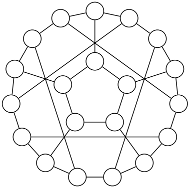

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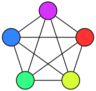
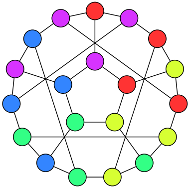


K_5



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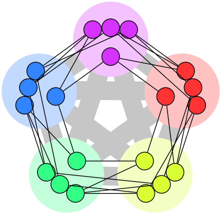

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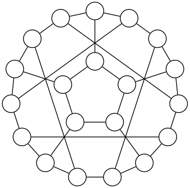

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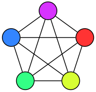

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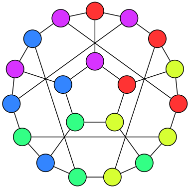

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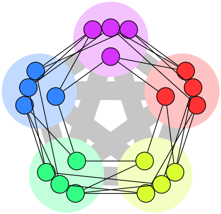

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Ex. $G \to K_k$ iff G is k-colorable.

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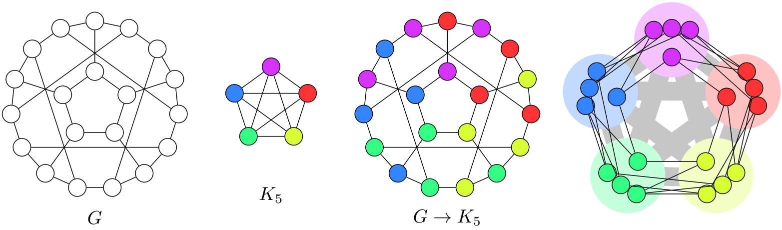

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PCSP(G,H):
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The conjecture for graph homomorphisms

PCSP(G,H) is hard? *conj.* Brakensiek, Guruswami '18
for all non-bipartite G, H such that $G \to H$.
It's enough to ask about PCSP(C_{2k+1},K_n).

$$\dots C_7 \to C_5 \to C_3 = K_3 \to K_4 \to K_5 \to \dots$$

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Results:

- PCSP(G,K_3) is NP-hard for all $G \to K_3$.
- This property of H that "for all G, PCSP(G,H) is NP-hard" only depends on the topology of H...

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The crucial gadget is $H^n = H \times H \times \cdots \times H$.
Possible H-colorings of the gadget are *polymorphisms* $H^n \to H$.
Often the only H-colorings of H^n are projections $p_i \colon H^n \to H$.
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If two gadgets are colored with $a \colon H^2 \to H$ and $b \colon H^5 \to H$,
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So for any function $\pi \colon \{1, \dots, 5\} \to \{1, 2\}$
we can enforce the constraint
"if the second gadget is colored p_i, then the first is colored $p_{\pi(i)}$".
"if we choose $i \in \{1, \dots, 5\}$, then we must choose $\pi(i) \in \{1, 2\}$".

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For $n \gg k$, a homomorphism $f\colon C_k^n \to C_3$ *looks* like a function that depends only on a few ($\sim k$ out of n) inputs, except for some noise.

Looking at it as a continuous function, we disregard the noise.

The box complex

Graphs	Topological spaces
G	$\mathrm{Box}(G)$
C_k	circle $\mathcal{S}^1$
K_k	sphere $\mathcal{S}^{k-2}$
C_k^n	torus $\mathcal{S}^1 \times \cdots \times \mathcal{S}^1 = (\mathcal{S}^1)^n$
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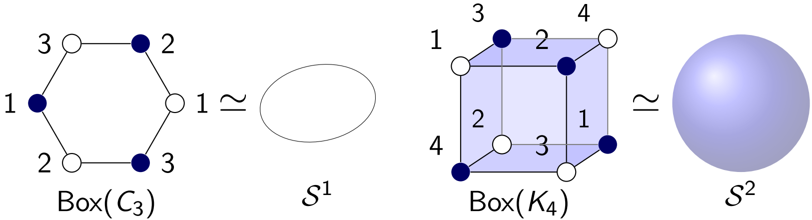

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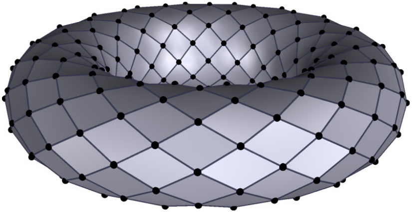

$\mathrm{Box}(C_k^2) \simeq (\mathcal{S}^1)^2$

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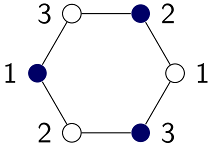

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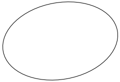
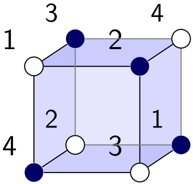


$\mathrm{Box}(C_3)$ $\qquad$ $\mathcal{S}^1$



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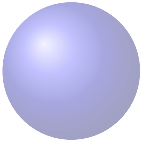
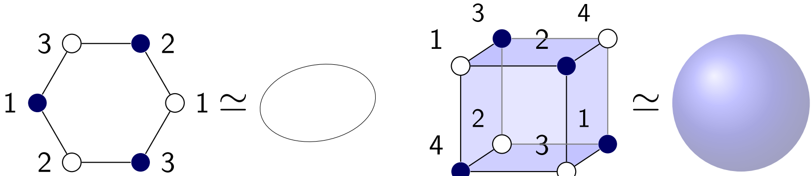


$\mathrm{Box}(K_4)$ $\qquad$ $\mathcal{S}^2$

The box complex

Graphs	Topological spaces **with an action of** $\mathbb{Z}_2$
G	$\mathrm{Box}(G)$
C_k	circle $\mathcal{S}^1$
K_k	sphere $\mathcal{S}^{k-2}$
C_k^n	torus $\mathcal{S}^1 \times \cdots \times \mathcal{S}^1 = (\mathcal{S}^1)^n$
$f\colon C_k^n \to C_3$	**equivariant** map from n-torus to circle



$\mathrm{Box}(C_3)$ $\qquad$ $\mathcal{S}^1$ $\qquad$ $\mathrm{Box}(K_4)$ $\qquad$ $\mathcal{S}^2$

$\mathrm{Pol}(C_k, C_3)$

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A map $\mathcal{S}^1 \to \mathcal{S}^1$ has a winding number $\deg(f) \in \mathbb{Z}$.
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There are finitely many possible $\deg((x, y) \mapsto f(x, y, y, x, \dots))$ (namely 3^{k^2}, independent of n).
So only a few $i \in \{1, \dots, n\}$ have non-zero degree when $n \gg k$!

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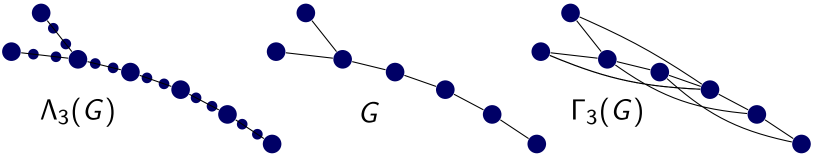
So someone has non-zero degree.
That's how we "decode" $f\colon C_k^n \to C_3$ to a small choice in $\{1, \dots, n\}$.

Adjunction

A graph *thin functor* Λ is a function from graphs to graphs such that:

$$G \to H \implies \Lambda G \to \Lambda H$$

For example: Λ_k replaces each edge with k edges
Γ_k puts an edge between endpoints of every k-walk



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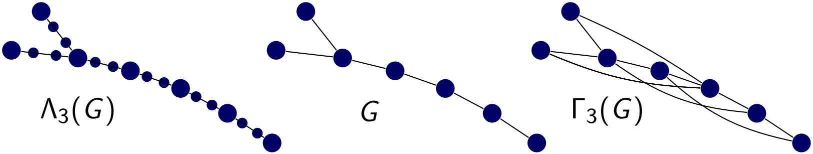
$$\Lambda G \to H \quad \Leftrightarrow \quad G \to \Gamma H$$



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Λ is a reduction from PCSP(G,ΓH) to PCSP(ΛG, H).
(So if we knew the first is hard, then the latter is hard).

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It turns out $\Omega_k G$ behaves like barycentric subdivision on $\mathrm{Box}(G)$:

- $\mathrm{Box}(\Omega_k G) \simeq \mathrm{Box}(G)$
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We use it to prove that "only topology matters":

if H is such that $\mathrm{PCSP}(G,H)$ is hard for all G
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Pf. Since H and H' have the same topologies, we have $\Omega_k H' \to H$.
By adjunction $\mathrm{PCSP}(\Gamma_k G,H')$ is harder than $\mathrm{PCSP}(G, \Omega_k H')$.
The latter is harder to get than $\mathrm{PCSP}(G, H)$.
So for cycles $\mathrm{PCSP}(\Gamma_k C_n,H') \geq \mathrm{PCSP}(C_n,H)$.
Since $\Gamma_k C_n \approx C_{n/k}$, increasing n proves hardness for large cycles.

Adjunction – some open problems

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But we know the functors we apply are not really adjoint, they change the minion in some interesting way!

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II. To understand $\mathrm{Pol}(C_k, K_4)$ we have to understand maps from n-torus to $\mathcal{S}^2$ (or rather from n-torus to $\mathbb{R}P^2$).

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II. To understand $\mathrm{Pol}(C_k, K_4)$ we have to understand maps from n-torus to $\mathcal{S}^2$ (or rather from n-torus to $\mathbb{R}P^2$).

III. When we look at $f \in \mathrm{Pol}(C_k, K_5)$ we have a problem:
the continuous functions we get from projections are all homotopic!
So $\mathrm{Box}(f)$ does not contain any interesting information.
But somehow $\mathrm{Box}(K_5)$ does?

Adjunction – some open problems

I. In category theory, the definition of adjoints has more conditions.
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Thank you!