Promise CSP: Arise, my minions!

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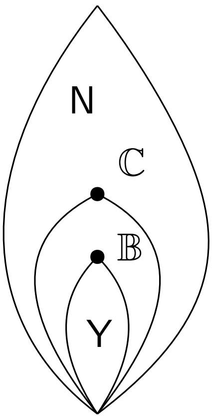
Promise CSP

- Constraint Satisfaction as a homomorphism problem
- Given $\mathbb{A}$, decide if $\mathbb{A} \to \mathbb{B}$
- CSP($\mathbb{B}$) is pretty well understood
- Promise CSP: Fix $\mathbb{B}$, $\mathbb{C}$ such that $\mathbb{B} \to \mathbb{C}$
- Input $\mathbb{A}$
- "Yes" instance when $\mathbb{A} \to \mathbb{B}$
- "No" instance when $\neg(\mathbb{A} \to \mathbb{C})$

Promise CSP in picture



- "Yes" instances below $\mathbb{B}$
- "No" instances not below $\mathbb{C}$
- Notice the gap!

Example: PCSP($\mathbb{K}_3, \mathbb{K}_n$)

- Structures: Graphs
- $\mathbb{G} \to \mathbb{K}_3$ if and only if $\mathbb{G}$ is 3-colorable
- $\mathbb{G} \to \mathbb{K}_n$ if and only if $\mathbb{G}$ is n-colorable
- PCSP($\mathbb{K}_3, \mathbb{K}_n$) has
- "Yes" instances 3-colorable
- "No" instances not even n-colorable
- Conjectured to be NP-hard

Why???

- How far can we push the dichotomy between P and NP-hard problems?
- Better understanding of CSP reductions
- Connections to approximability and things that CS people like
  1. Probabilistically Checkable Proofs (PCP)
  2. Label Cover problem
  3. Unique Games Conjecture
- New techniques (including category theory and topology; see future talks)

Polymorphisms from $\mathbb{A}$ to $\mathbb{B}$

- For CSP($\mathbb{A}$) we had polymorphisms: Mappings $\mathbb{A}^n \to \mathbb{A}$ that preserve relations
- Counterpart for PCSP($\mathbb{A}, \mathbb{B}$): Mappings $\mathbb{A}^n \to \mathbb{B}$ that preserve relations
- Denote this set by Pol($\mathbb{A}, \mathbb{B}$)
- First appearance as "weak polymorphisms": Per Austrin, Venkatesan Guruswami, and Johan Håstad. $(2 +$ epsilon)-SAT is NP-hard, 2014
- We can (in general) no longer compose polymorphisms (no longer a clone/algebra)
- What is Pol($\mathbb{A}, \mathbb{B}$) good for?

Calling all minions

- If $f(x_1, x_2, x_3)$ preserves relations, then so does $f(x_3, x_3, x_3)$
- In general, let $f\colon \mathbb{A}^n \to \mathbb{B}$ and $\sigma\colon [n] \to [m]$
- Define the σ-minor of f as

$$f^\sigma(x_1, \dots, x_m) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

- Example f ternary, $\sigma(1) = \sigma(2) = 14, \sigma(3) = 2$,

$$f^\sigma(x_1, \dots, x_{14}) = f(x_{14}, x_{14}, x_2)$$

- $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is nonempty and closed under minor-taking – a minion (AKA clonoid)

Minion homomorphisms

- Let $\mathcal{M}, \mathcal{N}$ be minions
- $\phi\colon \mathcal{M} \to \mathcal{N}$ is a minion homomorphism if it commutes with minor-taking: For all $f \in \mathcal{M}$ and all applicable σ

$$\phi(f^\sigma) = \phi(f)^\sigma.$$

- We do not have to worry about compositions!
- Compare to h1 clone homomorphisms in L. Barto, J. Opršal, M. Pinsker: The wonderland of reflections (2018)

A simple example from Tame Congruence Theory

- **A** algebra, e a unary operation from **A** with image $B \subset \mathbf{A}$
- For f term operation of **A** consider the mapping $f \mapsto e \circ f$
- This maps term operations of **A** into operations on B
- It is not an algebra homomorphism...
- ...but it is a clonoid homomorphism
- $e(f(x_3, x_2, x_2, x_3, x_7)) = e \circ f(x_3, x_2, x_2, x_3, x_7)$
- This would be a special case of “reflection” from the Wonderland paper

PCSP reduction

Theorem

If $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \to \mathrm{Pol}(\mathbb{C}, \mathbb{D})$*, then* $\mathrm{PCSP}(\mathbb{C}, \mathbb{D})$ *reduces to* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *in logarithmic space.*

- Libor Barto, Jakub Bulín, Andrei Krokhin, Jakub Opršal, Algebraic approach to promise constraint satisfaction
- In particular: If $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \to \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_3)$, then $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is NP-hard
- $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_3)$ contains only operations like $f(x_1, \dots, x_n) = \alpha(x_i)$
- Vladimír Müller, On colorings of graphs without short cycles, Discrete mathematics 26, 1979

Hardness of PCSP($\mathbb{K}_3, \mathbb{K}_4$)

- Original combinatorial proofs:
  1. Sanjeev Khanna, Nathan Linial, and Shmuel Safra. On the hardness of approximating the chromatic number, 2000
  2. Venkatesan Guruswami, Sanjeev Khanna, On the hardness of 3-coloring a 4-colorable graph, 2004
- Not state of the art anymore (see future talks)
- We want to find a homomorphism $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_4) \to \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_3)$

Coloring by projections

- We want to assign each $f\colon \mathbb{K}_3^n \to \mathbb{K}_4$ one of n coordinates so that we commute with minor-taking
- $\phi(f) = \pi_i$ should imply $\phi(f^\sigma) = \pi_{\sigma(i)}$
- Our job is easy: Each f in fact mostly depends on just one coordinate
- Proof modeled after Joshua Brakensiek, Venkatesan Guruswami, New hardness results for graph and hypergraph colorings, 2016

Lemma

Let $f\colon \mathbb{K}_3^n \to \mathbb{K}_4$ be a homomorphism. Then there exists $a \in V(\mathbb{K}_4)$ such that f restricted to $\mathbb{K}_3^n \setminus \{f^{-1}(a)\}$ depends only on one coordinate i. Moreover, this i is unique.

Example

- Condition for homomorphism $f\colon \mathbb{K}_3^2 \to \mathbb{K}_4$

$$f\begin{pmatrix} u & w \\ | & | \\ v & t \end{pmatrix} \in E(\mathbb{K}_4)$$

f	0	1	2
0	0	0	1
1	2	2	2
2	3	3	1

- Cross out all 1s. . .

Sketch of the general proof I

- Take $f\colon \mathbb{K}_3^n \to \mathbb{K}_4$
- View $V(\mathbb{K}_3^n)$ as $\mathbb{Z}_3^n$ for convenience
- Step 1: Show that there is no $\mathbf{v} \in V(K_3^n)$ and no distinct i, j such that

$$f(\mathbf{v}), \quad f(\mathbf{v}+\mathbf{e}_i), \quad f(\mathbf{v}+2\mathbf{e}_i)$$

  and

$$f(\mathbf{v}), \quad f(\mathbf{v}+\mathbf{e}_j), \quad f(\mathbf{v}+2\mathbf{e}_j)$$

  would contain three distinct values each.
- Proof by induction on n and considering a few cases.

Sketch of the general proof II

- Step 2: If there is $\mathbf{v}$ and i such that

$$f(\mathbf{v}) \neq f(\mathbf{v} + \mathbf{e}_i) = f(\mathbf{v} + 2\mathbf{e}_i),$$

  the claim holds.
- Say $f(00\dots0) = 0, f(10\dots0) = f(20\dots0) = 1$
- Then examine the cube $\{1,2\}^n$
- Observe that f on $\{1,2\}^n$ is 2 or 3
- Assume that $2 = f(11\mathbf{1}2222) \neq f(11\mathbf{2}2222) = 3$
- Then $3 = f(2221111)$ and $f(2211111) = 2$
- Thus $f(11\mathbf{0}2222) \in \{0,1\}$
- Aha, $f(11?2222)$ are all different!

Sketch of the general proof III

- Let $\mathbf{u}, \mathbf{w} \in \{1,2\}^n$ differ in one coordinate i
- We found: If $f(\mathbf{u}) \neq f(\mathbf{w})$ then $f(\mathbf{u}), f(\mathbf{u}+\mathbf{e}_i), f(\mathbf{u}+2\mathbf{e}_i)$ are all different
- By step 1 there is for each $\mathbf{u}$ at most one such i
- If the i exists, record it as $g(\mathbf{u})$
- Now say $f(111\mathbf{1}22) = f(111\mathbf{2}22) = 2$, but $g(111\mathbf{1}22) = 1$ and $g(111\mathbf{2}22)$ is not 1 (maybe undefined)
- Then $f(\mathbf{2}11\mathbf{1}22) = 3$ and $f(\mathbf{2}11\mathbf{2}22) = 2$
- Then $g(\mathbf{2}11\mathbf{1}22)$ is not unique

Sketch of the general proof IV

- For each $\mathbf{u} \in \{1,2\}^n$ either f is constant on all neighbors, or there is a well defined coordinate $g(\mathbf{u})$
- If $g(\mathbf{u})$ is defined, it spreads to neighbors
- f is not constant on $\{1,2\}^n$, so g is defined everywhere to be the same
- Considering a few cases gives us that f is “mostly” a projection to the g-th coordinate

Sketch of the general proof V

- So we know that f is in each direction constant or has 3 distinct values
- And for each $\mathbf{u}$ there is at most one direction so that f had 3 different values
- If the direction exists for $\mathbf{u}$, denote the corresponding coordinate by $g(\mathbf{u})$
- That's a lot of conditions on $f\dots$
- Again if $\mathbf{u}, \mathbf{w}$ are neighbors and $g(\mathbf{u})$ is defined, then $g(\mathbf{w}) = g(\mathbf{u})$
- By contradiction: Say $f(0000000) = 0, f(1000000) = 1, f(2000000) = 2$ and $0 = f(0000\mathbf{1}00) = f(1000\mathbf{1}00) = f(2000\mathbf{1}00)$
- Then $g(1000000) = 1$, so $f(1000100) = 1$, contradiction
- Thus f is the projection to the g-th coordinate

What's next?

- Studying minions for their own sake (the homomorphism order of minions is a distributive lattice!)
- Homomorphisms to minions where operations depend on small sets of coordinates
- Better hardness proofs, stronger than by minion homomorphisms
- Reductions between various $\mathrm{PCSP}(\mathbb{K}_n, \mathbb{K}_m)$ problems
- Different kinds of promises

New seminar website

http://math.colorado.edu/~alka3345/