A theory of gadget reductions for promise constraint satisfaction

Part II

Jakub Opršal

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Durham University

overview

Part I (last week)

- promise constraint satisfaction problems
- adjunctions give reductions between (P)CSPs
- gadget reductions (replacement) and pp-powers are adjoint.

Part II (today)

- describe the best gadget reduction
- show one more adjunction

previously...

Theorem. [Barto, Bulín, Krokhin, **O**, '19]
The following are equivalent for all pairs of similar relational structures $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{B}_1, \mathbf{B}_2$:

1. there is a gadget reduction from $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$;
2. $(\mathbf{B}_1, \mathbf{B}_2)$ is a homomorphic relaxation a pp-power of $(\mathbf{A}_1, \mathbf{A}_2)$;
3. *??!*

the best gadget reduction

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\Sigma_{\mathbf{B}_1}} \mathrm{PCSP}(\mathscr{P}, \mathscr{B}) \xrightarrow{\mathrm{id}} \mathrm{PCSP}(\mathscr{P}, \mathscr{A}) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

$\mathscr{A} = \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2), \mathscr{B} = \mathrm{pol}(\mathbf{B}_1, \mathbf{B}_2)$

$$\begin{aligned} \Sigma(\mathbf{A}, \mathbf{B}) \to \mathscr{M} \quad &\text{iff} \quad \mathbf{B} \to \mathbf{F}_{\mathscr{M}}(\mathbf{A}) \\ \mathbf{I}_{\mathbf{A}}(\Sigma) \to \mathbf{B} \quad &\text{iff} \quad \Sigma \to \mathrm{pol}(\mathbf{A}, \mathbf{B}) \end{aligned}$$

formulation of PCSP($\mathscr{M}$, $\mathscr{N}$)

Problem

Fix minions $\mathscr{M}$ and $\mathscr{N}$. Given a minor (strong Mal'cev) condition Σ,

- accept if $\Sigma \to \mathscr{M}$,
- reject if $\Sigma \not\to \mathscr{N}$.

A minion homomorphism is a mapping $\xi\colon \mathscr{M} \to \mathscr{N}$ s.t.

$$\xi(f)^{\pi} = \xi(f^{\pi}) \text{ for all } \pi\colon [n] \to [m].$$

Such homomorphisms preserve satisfaction of minor conditions.
($f^{\pi}(x_1, \dots, x_m) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$).

The function minion consisting of projections on a two-element set is denoted by $\mathscr{P}$. We have $\mathscr{P} \to \mathscr{M}$ for all minions $\mathscr{M}$.

$\mathbf{I}_{\mathbf{A}_1}\colon \mathrm{PCSP}(\mathscr{P}, \mathscr{M}) \to \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$

Given a minor condition Σ, construct an instance $\mathbf{I}_{\mathbf{A}_1}(\Sigma)$ of $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$:

- for each symbol f of arity n in Σ, take a copy of $\mathbf{A}_1^n$ with vertices labelled by $f(a_1, \dots, a_n)$ for $a_{1,\dots,n} \in \mathbf{A}_1$;
- for each identity

$$f(x_{\pi(1)}, \dots, x_{\pi(n)}) \approx g(x_1, \dots, x_m)$$

where $\pi\colon [n] \to [m]$, and $a_{1,\dots,m} \in \mathbf{A}_1$, identify vertices labelled

$$f(a_{\pi(1)}, \dots, a_{\pi(n)}) \text{ and } g(a_1, \dots, a_m).$$

adjoint to $\mathbf{I}_{\mathbf{A}_1}$: $\mathrm{pol}(\mathbf{A}_1, -)$

We say that $f\colon A_1^n \to A_2$ is a polymorphism from $\mathbf{A}_1$ to $\mathbf{A}_2$ of arity n if f is a homomorphism from $\mathbf{A}_1^n$ to $\mathbf{A}_2$.

The set of all such polymorphisms of arity n is denoted by $\mathrm{pol}^{(n)}(\mathbf{A}_1, \mathbf{A}_2)$, and $\mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2) = \bigcup_{n\in\mathbb{N}} \mathrm{pol}^{(n)}(\mathbf{A}_1, \mathbf{A}_2)$.

$\mathbf{I}$ & pol: the second reduction

Observation. For all $\mathbf{C}$, we have

$$\Sigma \to \mathrm{pol}(\mathbf{A}_1, \mathbf{C}) \iff \mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{C}.$$

Proof.
Assume $\xi\colon \Sigma \to \mathrm{pol}(\mathbf{A}_1, \mathbf{C})$ witnesses satisfcation of Σ. Define $h\colon \mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{C}$ by

$$h\colon f(a_1, \dots, a_n) \mapsto \xi(f)(a_1, \dots, a_n).$$

Observe that (1) h is well-defined, (2) h is a homomorphism.
For the other implication, assume a homomorphism $h\colon \mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{C}$, define ξ as

$$\xi(f)\colon (a_1, \dots, a_n) = h(f(a_1, \dots, a_n)).$$

∎

$\mathbf{I}$ & pol: the second reduction

Theorem

The indicator structure gives a reduction:

$$\mathrm{PCSP}(\mathscr{P}, \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

Proof. We have that $\mathbf{I}_{\mathbf{A}_1}$ is a reduction

$$\mathrm{PCSP}(\mathrm{pol}(\mathbf{A}_1, \mathbf{A}_1), \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)) \to \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

But $\mathscr{P} \to \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_1)$, so we get the required reduction by homomorphic relaxation.

Alternatively, we can show directly:

1. if Σ is trivial, then $\mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{A}_1$, and
2. if $\mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{A}_2$ then $\Sigma \to \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)$. ∎

the best gadget reduction

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\Sigma_{\mathbf{B}_1}} \mathrm{PCSP}(\mathscr{P}, \mathscr{B}) \xrightarrow{\mathrm{id}} \mathrm{PCSP}(\mathscr{P}, \mathscr{A}) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

$\mathscr{A} = \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2), \mathscr{B} = \mathrm{pol}(\mathbf{B}_1, \mathbf{B}_2)$

$$\Sigma(\mathbf{A}, \mathbf{B}) \to \mathscr{M} \quad \text{iff} \quad \mathbf{B} \to \mathbf{F}_{\mathscr{M}}(\mathbf{A})$$
$$\mathbf{I}_{\mathbf{A}}(\Sigma) \to \mathbf{B} \quad \text{iff} \quad \Sigma \to \mathrm{pol}(\mathbf{A}, \mathbf{B})$$

Σ: $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \to \mathrm{PCSP}(\mathscr{P}, \mathscr{B})$

Starting with $\mathbf{I}$ similar to $\mathbf{B}_1$, construct a minor condition $\Sigma(\mathbf{B}_1, \mathbf{I})$:

- for each $v \in I$, add to Σ a symbol f_v of arity B_1,
- for each $(v_1, \ldots, v_k) \in R^{\mathbf{I}}$, add to Σ a symbol $g_{(v_1,\ldots,v_k),R}$ of arity $R^{\mathbf{B}_1}$, and
- introduce identities

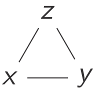
$$f_{v_1}(x_{b_1}, \ldots, x_{b_n}) \approx g_{(v_1,\ldots,v_k),R}(x_{r_1(1)}, \ldots, x_{r_m(1)})$$
$$\vdots$$
$$f_{v_k}(x_{b_1}, \ldots, x_{b_n}) \approx g_{(v_1,\ldots,v_k),R}(x_{r_1(k)}, \ldots, x_{r_m(k)})$$

where $R^{\mathbf{B}_1} = \{r_i \mid i \in [m]\}$ and $B_1 = \{b_i \mid i \in [n]\}$.

examples of conditions from structures

- $\Sigma(K_3, \circlearrowleft)$ is the Siggers identity!

$$
\begin{aligned}
v(x, y, z) &\approx s(x, y, z, x, y, z)\\
v(x, y, z) &\approx s(y, x, x, z, z, y)
\end{aligned}
$$



- $\Sigma(K_3, K_3)$ is trivial!
- $\Sigma(\text{1-in-3}, \circlearrowleft_3)$ is (non-idempotent) ternary weak near unanimity!
  (1-in-3 is the template of 1in3-Sat.)

adjoint to Σ: the free structure $\mathbf{F}$

Given a minion $\mathscr{M}$ and a (finite) structure $\mathbf{B}_1$, we define a structure $\mathbf{F}_{\mathscr{M}}(\mathbf{B}_1)$:

- the universe are the B_1-ary functions in $\mathscr{M}$, i.e., $F_{\mathscr{M}}(\mathbf{B}_1) = \mathscr{M}^{(B_1)}$,
- the relation $R^{\mathbf{F}}$ is defined to contain all tuples $(f_1, \dots, f_k)$ such that there is $g \in \mathscr{M}^{(R^{\mathbf{B}_1})}$ satisfying

$$f_1(x_{b_1}, \dots, x_{b_n}) \approx g(x_{r_1(1)}, \dots, x_{r_m(1)})$$

$$\vdots$$

$$f_k(x_{b_1}, \dots, x_{b_n}) \approx g(x_{r_1(k)}, \dots, x_{r_m(k)})$$

where $R^{\mathbf{B}_1} = \{r_i \mid i \in [m]\}$ and $B_1 = \{b_i \mid i \in [n]\}$.

example of a free structure

Example. The power structure [Feder, Vardi, "98] is the free structure of the semilattice clone.

Example. A variety is congruence permutable iff it has a Maltsev term [Maltsev, "54].

Proof. Consider

$$\mathbf{F}_{\mathrm{clo}\,\mathcal{V}}(\{x, y\}; B = \{(x, x), (x, y), (y, y)\})$$

Note that $\mathbb{B}^{\mathbf{F}} \in \mathcal{V}$, so the two kernels of projections permute which means

$$\exists q \in B^{\mathbf{F}} \text{ s.t. } y \approx q(x, x, y) \text{ and } x \approx q(x, y, y). \qquad \blacksquare$$

Σ & $\mathbf{F}$: the first reduction

Observation. for all $\mathbf{C}$, we have

$$\mathbf{C} \to \mathbf{F}_{\mathscr{M}}(\mathbf{B}_1) \iff \Sigma(\mathbf{B}_1, \mathbf{C}) \to \mathscr{M}$$

Theorem

The assignment $\mathbf{I} \mapsto \Sigma(\mathbf{B}_1, \mathbf{I})$ *gives a reduction:*

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\Sigma(\mathbf{B}_1, -)} \mathrm{PCSP}(\mathscr{P}, \mathrm{pol}(\mathbf{B}_1, \mathbf{B}_2))$$

the best gadget reduction

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\Sigma_{\mathbf{B}_1}} \mathrm{PCSP}(\mathscr{P}, \mathscr{B}) \xrightarrow{\mathrm{id}} \mathrm{PCSP}(\mathscr{P}, \mathscr{A}) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

$\mathscr{A} = \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2), \mathscr{B} = \mathrm{pol}(\mathbf{B}_1, \mathbf{B}_2)$

$$\Sigma(\mathbf{A}, \mathbf{B}) \to \mathscr{M} \quad \text{iff} \quad \mathbf{B} \to \mathbf{F}_{\mathscr{M}}(\mathbf{A})$$

$$\mathbf{I}_{\mathbf{A}}(\Sigma) \to \mathbf{B} \quad \text{iff} \quad \Sigma \to \mathrm{pol}(\mathbf{A}, \mathbf{B})$$

the best gadget reduction

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\Sigma_{\mathbf{B}_1}} \mathrm{PCSP}(\mathscr{P}, \mathscr{B}) \xrightarrow{\mathrm{id}} \mathrm{PCSP}(\mathscr{P}, \mathscr{A}) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

$\mathscr{A} = \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2), \mathscr{B} = \mathrm{pol}(\mathbf{B}_1, \mathbf{B}_2)$

To make the middle reduction work, we need

$$\mathscr{P} \to \mathscr{P} \quad \text{and} \quad \mathscr{A} \to \mathscr{B}.$$

Therefore, if $\mathscr{A} \to \mathscr{B}$, then $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ reduces to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$.

Theorem. $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ is log-space equivalent to $\mathrm{PCSP}(\mathscr{P}, \mathrm{pol}(\mathbf{A}, \mathbf{B}))$.

the best gadget reduction

Theorem
The discussed reduction is the best among gadget reductions.

Lemma [Wrochna, Živný]
If ρ preserves products, then there is a minion homomorphism

$$\mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2) \to \mathrm{pol}(\rho\mathbf{A}_1, \rho\mathbf{A}_2)$$

for all relational structures $\mathbf{A}_1, \mathbf{A}_2$.

Observation. For each gadget ϕ, ρ_ϕ preserves products. ■

an application

Goal. a reduction from $\mathrm{PCSP}(\mathbf{H}_2, \mathbf{H}_k)$ to $\mathrm{PCSP}(K_3, K_5)$.

$\mathbf{H}_k$ is the structure with domain $H_k = [k]$ and one ternary relation $\mathrm{nae}_k = [k]^3 \setminus \{(a, a, a) \mid a \in [k]\}$.

Theorem. [Dinur, Regev, Smyth, '05]
For all $k \geq 2$, $\mathrm{PCSP}(\mathbf{H}_2, \mathbf{H}_k)$ is NP-hard.

$$\mathrm{PCSP}(\mathbf{H}_2, \mathbf{F}_{\mathscr{K}_{3,5}}(\mathbf{H}_2)) \xrightarrow{\Sigma_{\mathbf{H}_2}} \mathrm{PCSP}(\mathscr{P}, \mathscr{K}_{3,5}) \xrightarrow{\mathbf{I}_{K_3}} \mathrm{PCSP}(K_3, K_5)$$

where $\mathscr{K}_{3,5} = \mathrm{pol}(K_3, K_5)$.

Need. $\mathbf{F}_{\mathscr{K}_{3,5}}(\mathbf{H}_2) \to \mathbf{H}_n$ for some n.

$\mathbf{F}_{\mathrm{pol}(K_3, K_5)}(\mathbf{H}_2)$

- vertices: $F = \mathrm{pol}^{(2)}(K_3, K_5)$,
- hyperedges: $(f_1, f_2, f_3) \in R^{\mathbf{F}}$ if $\exists g \in \mathrm{pol}^{(6)}(K_3, K_5)$ with

$$\begin{aligned} f_1(x, y) &\approx g(x, x, y, y, y, x) \\ f_2(x, y) &\approx g(x, y, x, y, x, y) \\ f_3(x, y) &\approx g(y, x, x, x, y, y). \end{aligned}$$

Claim. $\mathbf{F}_{\mathrm{pol}(K_3, K_5)}(\mathbf{H}_2) \to \mathbf{H}_n$ for some n.

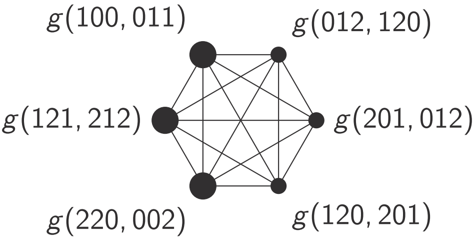
Since $\mathbf{F}$ is finite, it is enough to show that $\mathbf{F}$ does not have a 'hyperloop' (f, f, f). Such a hyperloop would give

$$g(x, x, y, y, y, x) \approx g(x, y, x, y, x, y) \approx g(y, x, x, x, y, y)$$

a.k.a. an Olšák polymorphism.

without Olšák things are hard

Proof. $\mathbf{I}_{\mathbf{K}_3}$(Olšák) contains:



$\blacksquare$

Corollary [Bulín, Krokhin, Opršal, '19]
For all $d \geq 3$, $\mathrm{PCSP}(K_d, K_{2d-1})$ is NP-hard.

Corollary
If pol$(\mathbf{A}, \mathbf{B})$ *contains no Olšák function, then* $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ *is NP-hard.*

beyond gadget reductions

the other adjoint to arc-graph [Wrochna, Živný, '20]

Reminder. The arc-graph ρG is the second pp-power defined by

$$(x_1, x_2) \in E \wedge (y_1, y_2) \in E \wedge x_2 = y_1.$$

- use the arc-graph pp-power as a reduction — this is the other way than you would expect!
- they obtain NP-hardness of PCSP$(K_k, K_{\binom{k}{\lfloor k/2 \rfloor}-1})$ for all $k \geq 4$.
- gives a reduction from PCSP(K_6, K_c) to PCSP$(K_4, K_{c'})$ which cannot be done by a gadget reduction.

the other adjoint to arc-graph [Wrochna, Živný, '20]

The right adjoint to the arc graph ωG is defined

- $V(\omega G) = \{(A^-, A^+) : A^\pm \subseteq V(G) \text{ and } A^- \times A^+ \subseteq E(G)\}$
- $E(\omega G) = \{((A^-, A^+), (B^-, B^+)) : A^+ \cap B^- \neq \emptyset\}$.

Theorem. gives a reduction from $\mathrm{PCSP}(K_6, K_c)$ to $\mathrm{PCSP}(K_4, K_{c'})$ which cannot be done by a gadget reduction.

Proof sketch.

- Need that $K_6 \to \omega K_4$.
- The vertices of graph ωK_4 are pairs of disjoint subsets of $[4]$.
- Fix the domain of K_6 to be $\binom{[4]}{2}$. Define

$$h\colon A \mapsto (A, [4] \setminus A).$$

- Observe that if $A \neq B$ then $A \cap ([4] \setminus B) \neq \emptyset$. ∎

conclusion

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\Sigma_{\mathbf{B}_1}} \mathrm{PCSP}(\mathscr{P}, \mathscr{B}) \xrightarrow{\mathrm{id}} \mathrm{PCSP}(\mathscr{P}, \mathscr{A}) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

- generalised loop conditions $\mathbf{C} \mapsto \Sigma(\mathbf{A}, \mathbf{C})$;
- free structure $\mathscr{M} \mapsto \mathbf{F}_{\mathscr{M}}(\mathbf{A})$;
- indicator structure $\Sigma \mapsto \mathbf{I}_{\mathbf{A}}(\Sigma)$;
- polymorphisms $\mathbf{C} \mapsto \mathrm{pol}(\mathbf{A}, \mathbf{C})$.

$$\Sigma(\mathbf{A}, \mathbf{B}) \to \mathscr{M} \quad \text{iff} \quad \mathbf{B} \to \mathbf{F}_{\mathscr{M}}(\mathbf{A})$$

$$\mathbf{I}_{\mathbf{A}}(\Sigma) \to \mathbf{B} \quad \text{iff} \quad \Sigma \to \mathrm{pol}(\mathbf{A}, \mathbf{B})$$

There are reductions beyond the algebraic approach!

credits

- pol-inv Galois correspondence [Pippenger, ‘02]
- polymorphisms in promise constraint satisfaction [Austrin, Håstad, Guruswami, ‘17]
- inclusions of function minions [Brakensiek, Guruswami, ‘18]
- h1 clone homomorphisms for CSPs [Barto, **O**, Pinsker, ‘18]
- minion homomorphisms [Barto, Bulín, Krokhin, **O**, ‘19]
- adjunctions [Wrochna, Živný, ‘20]

■