THIRD ASSIGNMENT

DUE MONDAY, OCTOBER 31

The following problems are in Chapter 15 of *Concise*, where there are some hints. Let $n \geq 1$ and π be an abelian group. In this problem, you will construct the Moore space $M(\pi, n)$ and the Eilenberg-MacLane space $K(\pi, n)$. You can use the Hurewicz Theorem.

(1) Construct a connected CW-complex $M(\pi, n)$ such that

$$\widetilde{H}_*(M(\pi, n), \mathbb{Z}) = \begin{cases} \pi & * = n \\ 0 & \text{otherwise.} \end{cases}$$

Solution 0.1. Let π have presentation

$$0 \to E_\pi \xrightarrow{\phi} F_\pi \to \pi \to 0$$

where $F_\pi = \mathbb{Z}\{s\}_{s \in S_\pi}$ and $E_\pi = \mathbb{Z}\{r\}_{r \in R_\pi}$ are free abelian groups. Consider the map

$$S_E = \bigvee_{R_\pi} S^n_r \xrightarrow{\Phi} \bigvee_{S_\pi} S^n_s = S_F$$

where for each $r \in R_\pi$, we define $\Phi : S^n \to \bigvee_{S_\pi} S^n$ as follows. Let $\phi(r) = \sum a_{r,s} s$ for $a_{r,s} \in \mathbb{Z}$. Then letting $S_r \subset S$ be those s's that appear in this sum,

$$\Phi|_{S^n_r} : S^n_r \xrightarrow{\nabla^{|S_r|-1}} \bigvee_{S_r} S^n \xrightarrow{\bigvee_{S_r} a_{r,s}} \bigvee_{S_r} S^n_s,$$

where $\nabla^k = (\nabla \vee \mathrm{id}) \circ \nabla^{k-1}$.

Let $M(\pi, n)$ be the homotopy cofiber of Φ. That is, it is the pushout:

$$\begin{tikzcd} \bigvee_{R_\pi} S^n_r \arrow[r, "\Phi"] \arrow[d] & \bigvee_{S_\pi} S^n_s \arrow[d] \\ \bigvee_{R_\pi} D^{n+1}_r \arrow[r] & M(\pi, n). \end{tikzcd}$$

Therefore, $M(\pi, n)^q = *$ for $0 \leq q < n$, $M(\pi, n)^n = \bigvee_{S_\pi} S^n_s$ with the standard cell structure with one n-cell for each sphere in the wedge and $M(\pi, n)^{n+1} = M(\pi, n)$.

Then, there is an exact sequence on homology

$$\widetilde{H}_{q+1}(M(\pi, n)) \to \widetilde{H}_q(S_E) \to \widetilde{H}_q(S_F) \to \widetilde{H}_q(M(\pi, n)) \to \widetilde{H}_{q-1}(S_E)$$

Since S_E and S_F are wedges of spheres, we get that $\widetilde{H}_q(S_E) = \widetilde{H}_q(S_F) = 0$ if $q \neq n$. Hence, $\widetilde{H}_q(M(\pi, n)) = 0$ if $q \neq n, n+1$ and we have a commutative diagram with top row exact:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_{n+1}(M(\pi,n)) \arrow[r] & \widetilde{H}_n(S_E) \arrow[r, "\widetilde{H}_*(\Phi)"] \arrow[d, equal] & \widetilde{H}_n(S_F) \arrow[r] \arrow[d, equal] & \widetilde{H}_n(M(\pi,n)) \arrow[r] & 0 \\
& & \bigoplus_{s\in S_\pi} \mathbb{Z} \arrow[r, "\phi"] & \bigoplus_{r\in R_\pi} \mathbb{Z} & &
\end{tikzcd}$$

Since ϕ is injective, $\widetilde{H}_{n+1}(M(\pi, n)) = 0$, and hence, $\widetilde{H}_n(M(\pi, n)) \cong \pi$.

(2) Give an example of an $M(\mathbb{Z}/2, 1)$ and of a $M(\mathbb{Z}, 2)$.

Solution 0.2. $\mathbb{R}P^2$ is an $M(\mathbb{Z}/2, 1)$ and $\mathbb{C}P^1 = S^2$ is an $M(\mathbb{Z}, 2)$. In general, S^n is an $M(\mathbb{Z}, n)$.

(3) (Chapter 15, Problem 4) Construct a connected CW-complex $K(\pi, n)$ such that

$$\pi_*(K(\pi,n),\mathbb{Z}) = \begin{cases} \pi & * = n \\ 0 & \text{otherwise.} \end{cases}$$

Solution 0.3. First, note that $\pi_q M(\pi, n) = 0$ if $q < n$. Indeed, the q–skeleton $M(\pi, n)^q = *$, so by cellular approximation, any map $S^q \to M(\pi, n)$ factors trough a point, hence is trivial. Further, by the Hurewicz theorem, using the fact that π is abelian if $n = 1$, we have that $\pi_n M(\pi, n) \cong \widetilde{H}_n(M(\pi, n)) \cong \pi$.

In Lemma 2.1 of Class20.pdf, let $X = Y_0 = M(\pi, n)$ and $Y = Y_1$. Then as shown there,

$$\pi_q Y_1 \cong \begin{cases} 0 & 0 \leq q < n \\ \pi & q = n \\ 0 & q = n+1. \end{cases}$$

Note that Y_1 is built from $M(\pi, n)$ by attaching $n+2$–cells. Therefore, Y_1 is a CW complex of dimension $n + 2$ with $Y_1^q = M(\pi, n)^q$ for $q \leq n + 1$.

Applying this lemma inductively, we obtain a space Y_k such that

$$\pi_q Y_k \cong \begin{cases} 0 & 0 \leq q < n \\ \pi & q = n \\ 0 & q \leq n+k. \end{cases}$$

and such that Y_k is a CW-complex of dimension $n + k + 1$ with $Y_k^q = Y_{k-1}^q$ for $q \leq n + k$.

Let $K(\pi,n) = \operatorname{colim}_k Y_k$. Then,

$$K(\pi,n)^q = \begin{cases} M(\pi,n) & q \le n+1 \\ Y_k & q = n+k+1. \end{cases}$$

Further, $\pi_q(K(\pi,n)) = \pi_q K(\pi,n)^{q+1}$, so that $K(\pi,n)$ has the required homotopy groups.

(4) Let X be another CW-complex X whose only non-zero homotopy group is $\pi_n X = \pi$. Construct a homotopy equivalence $K(\pi,n) \to X$. Conclude that $K(\pi,n)$'s are unique up to a weak homotopy equivalence.

Solution 0.4. Fix an isomorphism $\pi \xrightarrow{f} \pi_n X$ For each $s \in S_\pi$, let $S^n_s \xrightarrow{f_s} X$ be a representative of $f(\phi(s)) \in \pi_n X$. This gives a map

$$\bigvee_{S_\pi} S^n_s \xrightarrow{f = \bigvee f_s} X.$$

Then, applying $\widetilde{H}_n$, we get a commutative diagram

$$\begin{tikzcd} \bigoplus_{R_\pi} \widetilde{H}_n(S^n_r) \arrow[r, "\widetilde{H}_n(\Phi)"] & \bigoplus_{S_\pi} \widetilde{H}_n(S^n_s) \arrow[r, "\bigoplus \widetilde{H}_n(f_s)"] \arrow[d, equal] & \widetilde{H}_n X \\ & \pi \arrow[ur, dotted, "f"] & \end{tikzcd}$$

By the Hurewicz isomormophism, we have the same diagram if we replace $\widetilde{H}_n$ by π_n so that $f \circ \Phi \simeq *$. Hence, we get a diagram, which commutes up to homotopy.

$$\begin{tikzcd} S_E \arrow[r] & S_F \arrow[r, "f"] \arrow[d] & X \\ & M(\pi,n) \arrow[ur, "\psi_0"] & \end{tikzcd}$$

By construction, ψ_0 is an isomorphism on π_q for $q \le n$.

Now, for $k \geq 0$, we construct ψ_k inductively using the fact that any map from $S^{n+k+1} \to X$ extends the disk since $\pi_q X = 0$ if $q > n$:

$$
\begin{tikzcd}
\bigvee S^{n+k+1} \arrow[r] \arrow[d] & Y_k \arrow[d] \arrow[ddr, bend left, "\psi_k"] & \\
\bigvee D^{n+k+2} \arrow[r] \arrow[drr, dotted, bend right] & Y_{k+1} \arrow[dr, dotted, "\psi_{k+1}"'] & \\
 & & X
\end{tikzcd}
$$

The universal property of the colimit gives a map $K(\pi, n) \xrightarrow{\psi} X$. The fact that $\pi_q \psi$ is an isomorphism if $q \neq n$ is trivial. If $q = n$, using the fact that $K(\pi, n)^{n+1} = M(\pi, n)$, we have a commutative diagram

$$
\begin{tikzcd}
\pi_n M(\pi, n) \arrow[r, "\pi_n \psi_0"] \arrow[d, "\cong"'] & \pi_n X \\
\pi_n K(\pi, n) \arrow[ur, "\psi"] &
\end{tikzcd}
$$

Since $\pi_n \psi_0$ is an isomorphism, ψ is a weak homotopy equivalence.

Since $K(\pi, n)$ and X are CW-complexes, Whitehead's theorem implies that the are homotopy equivalence. Now, consider any space Y such that $\pi_n Y = \pi$ and $\pi_q Y = 0$ if $q \neq n$. Take a CW-approximation $X \to Y$. Then

$$K(\pi, n) \xrightarrow{\psi} X \to Y$$

is a weak equivalence.

(5) Exhibit a fibration $F \to E \to B$ where, up to weak homotopy equivalence, F is a $K(\pi, n-1)$, B is a $K(\pi, n)$ and E is contractible.

Solution 0.5. Consider the fibration

$$\Omega K(\pi, n) \to PK(\pi, n) \to K(\pi, n)$$

and note that $PK(\pi, n)$ is contractible, and $\pi_q \Omega K(\pi, n) \cong \pi_{q+1} K(\pi, n)$.