# SECOND ASSIGNMENT - SOLUTIONS

(1) Suppose that $E \xrightarrow{p} B$ is a fibration of unbased spaces with $B$ path connected. For $b \in B$, let $F_b = p^{-1}(b)$. Prove that $F_{b_1}$ is homotopy equivalent to $F_{b_2}$ for all $b_1, b_2 \in B$.

**Solution.** (See *Concise* - Chapter 7.6 for another solution to this problem.) We have shown that the inclusion $F_b \to P_{f,b}$, where

$$P_{f,b} = \{(e, \alpha) \mid \alpha(0) = b, \alpha(1) = e\}$$

is a homotopy equivalence for all $b$. Hence, it suffices to show that $P_{f,b_1} \simeq P_{f,b_2}$. Let

$$\gamma : I \to B$$

be a path from $b_1$ to $b_2$. Let

$$\Phi_{b_1}^{b_2} : P_{f,b_1} \to P_{f,b_2}$$

where $\Phi_{b_1}^{b_2}(e, \alpha) = (e, \gamma^{-1} * \alpha)$, with $*$ the usual composition of loops. Let

$$\Phi_{b_2}^{b_1} : P_{f,b_2} \to P_{f,b_1}$$

where $\Phi_{b_2}^{b_1}(e, \alpha) = (e, \gamma * \alpha)$. Then,

$$\Phi_{b_1}^{b_2} \circ \Phi_{b_2}^{b_1} : P_{f,b_2} \to P_{f,b_2}$$

is given by

$$\Phi_{b_1}^{b_2} \circ \Phi_{b_2}^{b_1}(e, \alpha) = (e, \gamma^{-1} * (\gamma * \alpha)).$$

Let

$$PB_{b_2} = \{\alpha \in B^I \mid \alpha(0) = b_2\}.$$

Let $\pi : I \to I$ be a reparametrization of $I$ such that $\sigma^{-1} * (\sigma * \beta)(\pi(t)) = \beta$ for any path $\beta$. Then Then,

$$H : P_{f,b_2} \times I \to P_{f,b_2}$$

given by $H(e, \alpha, s) = (e, \gamma^{-1} * (\gamma * \alpha)((1-s)\pi(t) + ts))$ is a homotopy between $\Phi_{b_1}^{b_2} \circ \Phi_{b_2}^{b_1}$ and the identity. A similar proof shows that $\Phi_{b_2}^{b_1} \circ \Phi_{b_1}^{b_2} \simeq \mathrm{id}$.

(2) Prove that there are homeomorphisms $\Sigma C_f \cong C_{\Sigma f} \cong C_{-\Sigma f}$, where $C_f$ here denotes the reduced mapping cone.

**Solution.** So we first prove that $C_{\Sigma f} \cong \Sigma C_f$. We have the following diagram

$$\begin{tikzcd}
\Sigma X \arrow[r, "\mathrm{id}"] \arrow[d, "i_0"] & \Sigma X \arrow[r, "\Sigma f"] \arrow[d, "\Sigma i_0"] & \Sigma Y \arrow[d] \\
C\Sigma X \arrow[r, "g"] & \Sigma C X \arrow[r] & \Sigma C_f
\end{tikzcd}$$

where $g((x \wedge t), s) = g((x, s) \wedge t)$. The right hand square is a pushout: This proof is similar to the proof that $(X \cup_f Y) \times I \cong (X \times I) \cup_{f \times \mathrm{id}} (Y \times I)$. Since $g$ is a homeomorphism, the pushout of the outer square is the same as that of the right hand square.

For the second part, note first that $C_{-\Sigma f} \cong C_{\Sigma f}$. Indeed, using the fact that $\tau^2 = \mathrm{id}$ on the nose, the following diagram gives a homeomorphism $C_{\Sigma f} \to C_{-\Sigma f}$.

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma f"] \arrow[d] & \Sigma Y \arrow[d] \arrow[ddrr, leftrightarrow, bend left, "\tau"] & & \\
C\Sigma X \arrow[r] \arrow[ddrr, bend right, "\mathrm{id}"'] & C_{\Sigma f} \arrow[ddrr, leftrightarrow, dotted] & & \\
& & {}^{\tau}\Sigma X \arrow[r, "-\Sigma f"] \arrow[d] & \Sigma Y \arrow[d] \\
& & C\Sigma X \arrow[r] & C_{-\Sigma f}
\end{tikzcd}$$

(3) If $p : E \to B$ is a fibration and $B$ is contractible, prove that there is a homotopy equivalence $\phi : E \to B \times F$ such that the following diagram commutes

$$\begin{tikzcd}
E \arrow[rr, "\phi"] \arrow[dr, "p"'] & & B \times F \arrow[dl, "\pi_B"] \\
& B &
\end{tikzcd}$$

(see 4.3.18 - Aguilar et al. for a hint.)

**Solution.** (Note, this is simpler if one assumes that $B$ has as base point preserving contraction. Also, I'm a certain there is a more elegant solution than the one I am giving below, but this is in the line with the hint in Aguilar et al.) Let $H : B \times I \to B$ be a contracting homotopy. For each $b \in B$, the map $H$ gives a path $\widehat{H}(b) : \{b\} \times I \to B$ from $b$ to the base point $*$.

Let

$$E_p = \{e, \alpha \mid \alpha(0) = p(e)\}.$$

Note that $(e, \widehat{H}(p(e))) \in E_p$. Since $p : E \to B$ is a fibration, there is a continuous map $s : E_p \to E^I$ such that

$$\Gamma(e, \alpha)(0) = e$$

$$p_*(\Gamma(e, \alpha)) = \alpha.$$

In particular, $p_*(\Gamma(e, \widehat{H}(p(e)))(1)) = \widehat{H}(p(e))(1) = *$, so $\Gamma(e, \widehat{H}(p(e)))(1) \in F$.

Let $\phi : E \to B \times F$ be given by

$$\phi(e) = (p(e), \Gamma(e, \widehat{H}(p(e)))(1))$$

and $\psi : B \times F \to E$ be given by

$$\psi(b, f) = \Gamma(f, \widehat{H}(b)^{-1})(1)$$

First, consider $\psi \circ \phi : E \to E$ given by

$$\psi \circ \phi(e) = \Gamma(\Gamma(e, \widehat{H}(p(e)))(1), \widehat{H}(p(e))^{-1})(1).$$

Let

$$\Psi : E \times I \to E$$

be given by

$$\Psi(e, t) = \begin{cases} \Gamma(\Gamma(e, \widehat{H}(p(e)))(1), \widehat{H}(p(e))^{-1})(1 - 2t) & 0 \leq t \leq 1/2 \\ \Gamma(e, \widehat{H}(p(e)))(2(1 - t)) & 1/2 \leq t \leq 1. \end{cases}$$

Then, $\Psi$ is a homotopy from $\psi \circ \phi$ to the identity on $E$.

Next, we will show that $\phi \circ \psi : B \times F \to B \times F$, i.e.

$$\phi \circ \psi(b, f) = \phi(s(f, \widehat{H}(b)^{-1})(1)) = (b, \Gamma(\Gamma(f, \widehat{H}(b)^{-1})(1), \widehat{H}(b))(1))$$

is homotopic to the identity. Let $c_*$ be the constant path in $B$ at $*$. For each point $f \in F$, consider the path

$$\sigma_f(t) = \begin{cases} \Gamma(f, c_*)(2t) & 0 \leq t \leq 1/2 \\ \Gamma(\Gamma(f, c_*)(1), c_*)(2t - 1) & 1/2 \leq t \leq 1. \end{cases}$$

Note that $\sigma_f : I \to F$ and that $\sigma_f(0) = f$ and $\sigma_f(1) = \Gamma(\Gamma(f, c_*)(1), c_*)(1)$.

Let $I = [0,2]$ and
$$\Phi : B \times F \times I \to B \times F$$
be given by
$$\Phi(b,f,s) = (b, \Gamma(\Gamma(f, \widehat{H}(b)^{-1}(t(1-s)))(1), \widehat{H}(b)^{-1}((1-t)(1-s)))(1))$$
for $s \in [0,1]$ and
$$\Phi(b,f,s) = (b, \sigma_f(2-s))$$
for $s \in [1,2]$. Then,
$$\Phi(b,f,0) = \phi \circ \psi(b,f)$$
and
$$\Phi(b,f,1) = (b, \Gamma(\Gamma(f, c_*)(1), c_*)(1))$$
and
$$\Phi(b,f,2) = (b,f).$$
So, $\Phi$ is a homotopy from $\phi \circ \psi$ to the identity.

(4) (a) Compute $\pi_* S^1$.

**Solution.** Since $S^1$ is connected, $\pi_0 S^1 = 0$. The universal cover of $S^1$ is $\mathbb{R} \to S^1$ with fibers $\mathbb{Z}$ as a discrete topological space. Therefore, $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_n S^1 = \pi_n \mathbb{R} = 0$ otherwise.

(b) Use the Hopf fibration $S^1 \to S^3 \to S^2$ to compute $\pi_3 S^2$.

**Solution.** There is an exact sequence
$$0 = \pi_2 S^3 \to \pi_2 S^2 \to \pi_1 S^1 \to \pi_1 S^3 = 0$$
so $\pi_2 S^2 \cong \mathbb{Z}$. This is in the range of the suspension isomorphism so $\pi_3 S^3 \cong \pi_{2+1} \Sigma S^2 \cong \mathbb{Z}$. Further, there is an exact sequence
$$0 = \pi_3 S^1 \to \pi_3 S^3 \to \pi_3 S^2 \to \pi_2 S^1 = 0$$
so $\pi_3 S^2 \cong \mathbb{Z}$.

(c) Compute $\pi_* \mathbb{C}P^\infty$. (Hint: there are fiber bundles $S^{2n+1} \to \mathbb{C}P^n$ with fiber $S^1$. Can you write $\mathbb{C}P^\infty$ as the based space in a fiber bundle?).

**Solution.** $\mathbb{C}P^\infty$ is connected so $\pi_0 \mathbb{C}P^\infty = 0$. There is a fiber sequence $S^1 \to S^\infty \to \mathbb{C}P^\infty$. Since $S^\infty \simeq *$, $\pi_n \mathbb{C}P^\infty \cong \pi_{n-1} S^1$. Hence, $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$. Alternatively, one can use the fibrations $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ and note that $(\mathbb{C}P^\infty)^{2n} = \mathbb{C}P^n$ so that $\pi_n(\mathbb{C}P^\infty) = \pi_n(\mathbb{C}P^n)$ for $n \geq 1$.