FIRST ASSIGNMENT

DUE MONDAY, SEPTEMBER 19

(1) Let $E \subset X \times X$ be an equivalence relation on a set X. Construct the set of equivalence classes as colimit in the category Sets.

(2) Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

be a commutative diagram.

(a) Prove that if the two inner squares are pushouts, then so is the outer rectangle. That is, suppose that both $Y = B \sqcup_A X$ and $Z = C \sqcup_B Y$. Prove that $Z = C \sqcup_A X$.

(b) What about if $Y = B \sqcup_A X$ and $Z = C \sqcup_A X$, then is $Z = C \sqcup_B Y$? And what if $Z = C \sqcup_B Y$ and $Z = C \sqcup_A X$, is $Y = B \sqcup_A X$?

(3) In the following problem, let $S^{n-1} \to D^n$ be the inclusion of the boundary, $X \vee X \to X \times X$ be the map $(\mathrm{id} \times *) \vee (* \times \mathrm{id})$ and $X \vee X \to X$ be the fold map $\mathrm{id} \vee \mathrm{id}$. For a based topological space X, let $J_2(X) = (X \times X)/((x, *) \sim (*, x))$. That is, $J_2(X)$ is the push-out

$$\begin{tikzcd} X \vee X \arrow[r] \arrow[d] & X \arrow[d] \\ X \times X \arrow[r] & J_2(X). \end{tikzcd}$$

Further, you may use the fact that $D^{2n} \cong I^{2n} \cong D^n \times D^n$ and other such standard homeomorphisms without proof.

(a) Describe $S^n \times S^n$ as a CW-complex obtained from $S^n \vee S^n$ by attaching a single $2n$–cell. Exhibit this as a pushout.

(b) Use your construction in (b) to give $J_2(S^n)$ the structure of a CW-complex with one n–cell and one $2n$–cell.

(c) Show that S^n is an H-space if and only if the attaching map of the $2n$-cell of $J_2(S^n)$ is null-homotopic.

(4) Let X be 1–connected (i.e., $\pi_0 X = \pi_1 X = 0$). Recall that for a covering map $p : E \to B$ of based spaces, given a base point preserving map $f : X \to B$, there exists a unique base point preserving lift $\widetilde{f} : X \to E$ such that $p\widetilde{f} = f$.

(a) Prove that $\pi_n p : \pi_n E \to \pi_n B$ is an isomorphism for $n \geq 2$.

(b) Compute $\pi_k \mathbb{R}P^n$ in terms of $\pi_k S^n$. What about $\pi_k \mathbb{R}P^\infty$?