FIRST ASSIGNMENT

DUE MONDAY, SEPTEMBER 19

(1) Let $E \subset X \times X$ be an equivalence relation on a set X. Construct the set of equivalence classes as colimit in the category Sets.

Solution. Let $\overline{X} = \{[x] \mid x \in X\}$ be the set of equivalence classes and $q : X \to \overline{X}$ be the quotient. Let $p_1, p_2 : E \to X$ be the two projections. We show that

$$E \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{q} \overline{X}$$

is a coequilizer diagram. Then Y is isomorphic to the set of equivalence classes. A map $X \to Z$ satisfies $f(p_1(x, y)) = f(p_2(x, y))$ if and only if $f(x) = f(y)$ if $(x, y) \in E$. Define $f' : \overline{X} \to Z$ by $f'([x]) = f(y)$ for any choice of $y \in [x]$. This is well defined since $y \in [x]$ implies that $(x, y) \in E$ so $f(x) = f(y)$. Further, $f'(q(x)) = f'([x]) = f(x)$ so $f' \circ q = f$. Let $f'' : \overline{X} \to Z$ be any other map such that $f'' \circ q = f$. Then, for $[x] \in \overline{X}$, $f''([x]) = f(q(x)) = f'([x])$ so $f' = f''$ and the hence f' is unique defined. Therefore, $\overline{X}$

(2) Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

be a commutative diagram.

(a) Prove that if the two inner squares are pushouts, then so is the outer rectangle. That is, suppose that both $Y = B \sqcup_A X$ and $Z = C \sqcup_B Y$. Prove that $Z = C \sqcup_A X$.

Solution. Given maps $X \to W$ and $C \to W$ making

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr] & \\ X \arrow[r] \arrow[drrr] & Y \arrow[r] & Z & \\ & & & W \end{tikzcd}$$

commute, since the left hand square is a pull back, we get a unique map $Y \to W$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr] & \\
X \arrow[r] \arrow[drrr] & Y \arrow[r] \arrow[drr, dotted] & Z & \\
 & & & W
\end{tikzcd}$$

making the diagram commute. Now, since the right hand square is a pull-back, we obtain a unique map $Z \to W$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr] & \\
X \arrow[r] \arrow[drrr] & Y \arrow[r] \arrow[drr] & Z \arrow[dr, dotted] & \\
 & & & W
\end{tikzcd}$$

making the diagram commute. Therefore, Z has the required universal property.

(b) What about if $Y = B \sqcup_A X$ and $Z = C \sqcup_A X$, then is $Z = C \sqcup_B Y$? And what if $Z = C \sqcup_B Y$ and $Z = C \sqcup_A X$, is $Y = B \sqcup_A X$?

Solution. If $Y = B \sqcup_A X$ and $Z = C \sqcup_A X$, then $Z = C \sqcup_B Y$. Indeed, suppose that we have maps

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr] & \\
X \arrow[r] & Y \arrow[r] \arrow[drr] & Z & \\
 & & & W
\end{tikzcd}$$

making the right hand diagram commute. Then the composites $A \to X \to Y \to W$ and $A \to B \to C \to W$ agree, and since the outer rectangle is a pushout, we obtain a unique map $Z \to W$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr] & \\
X \arrow[r] & Y \arrow[r] \arrow[drr, "f"'] & Z \arrow[dr, dotted] & \\
& & & W
\end{tikzcd}$$

such that $X \to Y \to W$ is equal to $X \to Y \to Z \to W$ and $C \to W$ is equal to $C \to Z \to W$. We need to prove that $Y \to Z \to W$ is equal to $Y \to W$. However, both these maps fulfill the universal property for the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr] & \\
X \arrow[r] \arrow[drr] & Y & \\
& & W
\end{tikzcd}$$

for the maps $B \to C \to W$ and $X \to Y \xrightarrow{f} W = X \to Y \to Z \to W$. Since Y is the pushout, the two maps must be equal.
On the other hand, if $Z = C \sqcup_B Y$ and $Z = C \sqcup_A X$, then it is not necessarily the case that $Y = B \sqcup_A X$. An counter-example in sets is given by

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d] & \{1,2\} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\
\{1\} \arrow[r] & \{1\} \arrow[r] & \{1\}
\end{tikzcd}$$

where the maps are the inclusions when there is a choice.

(3) In the following problem, let $S^{n-1} \to D^n$ be the inclusion of the boundary, $X \vee X \to X \times X$ be the map $(\mathrm{id} \times *) \vee (* \times \mathrm{id})$ and $X \vee X \to X$ be the fold map $\mathrm{id} \vee \mathrm{id}$. For a based topological space X, let $J_2(X) = (X \times X)/((x,*) \sim (*,x))$. That is, $J_2(X)$ is the push-out

$$\begin{tikzcd}
X \vee X \arrow[r] \arrow[d] & X \arrow[d] \\
X \times X \arrow[r] & J_2(X).
\end{tikzcd}$$

Further, you may use the fact that $D^{2n} \cong I^{2n} \cong D^n \times D^n$ and other such standard homeomorphisms without proof.

(a) Describe $S^n \times S^n$ as a CW-complex obtained from $S^n \vee S^n$ by attaching a single $2n$–cell. Exhibit this as a pushout.

Solution. We first look at the more general situation where we have $A \subset X$ and $B \subset Y$ with quotient maps both denoted by $q : X \to X/A$ and $q : Y \to Y/B$. We prove that

$$\begin{tikzcd}
X \times B \cup A \times Y \arrow[r, "(q\times *)\vee(*\times q)"] \arrow[d] & X/A \vee Y/B \arrow[d, "(1\times *)\vee(*\times 1)"'] \arrow[ddr, bend left, "F"] & \\
X \times Y \arrow[r, "q\times q"] \arrow[drr, bend right, "G"'] & X/A \times Y/B \arrow[dr] & \\
& & Z
\end{tikzcd}$$

is a pushout. Suppose we are given maps $F : X \times Y \to Z$ and $G : X/A \vee Y/B \to Z$ making the diagram commute. Define

$$H : X/A \times Y/B \to Z$$

by

$$H(\overline{x}, \overline{y}) = G(x, y).$$

Then, for any $a \in A$,

$$H(\overline{a}, \overline{y}) = G(a, y) = F(\overline{y})$$

and for any $b \in B$,

$$H(\overline{x}, \overline{b}) = G(x, b) = F(\overline{x})$$

so this is well-defined. It is also continuous since for any open $U \subset Z$,

$$H^{-1}(U) = (q \times q)(G^{-1}(U))$$

Further, this map is uniquely defined since $q \times q$ is surjective.

Now, consider $(X, A) = (Y, B) = (D^n, \partial D^n)$. The previous construction gives a pushout

$$\begin{tikzcd}
D^n \times \partial D^n \cup \partial D^n \times D^n \arrow[r] \arrow[d] & D^n/\partial D^n \vee D^n/\partial D^n \arrow[d] \\
D^n \times D^n \arrow[r] & D^n/\partial D^n \times D^n/\partial D^n.
\end{tikzcd}$$

Identifying $\partial(D^n \times D^n) = D^n \times \partial D^n \cup \partial D^n \times D^n$, $D^n/\partial D^n \cong S^n$ and $D^n \times D^n \cong D^{2n}$ proves the claim.

(b) Use your construction in (a) to give $J_2(S^n)$ the structure of a CW-complex with one n–cell and one $2n$–cell.

Solution. Consider the following commutative diagram, where the right hand square is the pushout defining $J_2(S^n)$ and the left hand square is the pushout of part (a).

$$\begin{tikzcd}
\partial D^{2n} \arrow[r] \arrow[d] & S^n \vee S^n \arrow[r] \arrow[d] & S^n \arrow[d] \\
D^{2n} \arrow[r] & S^n \times S^n \arrow[r] & J_2(S^n).
\end{tikzcd}$$

By 2(a), the outer square is a pushout diagram, which proves the claim.

(c) Show that S^n is an H-space if and only if the attaching map of the $2n$-cell of $J_2(S^n)$ is null-homotopic.

Solution. First, note that if $S^n \times S^n \xrightarrow{\mu} S^n$ gives S^n the structure of an H–space, then there are homotopies $H' : S^n \times I \to S^n$ from $\mu \circ (1 \times *)$ to the identity and $H'' : S^n \times I \to S^n$ from $\mu \circ (* \times 1)$ to the identity. Letting

$$H : (S^n \vee S^n) \times I \to S^n$$

be given by $H'(x, t)$ if x is in the left factor and $H''(x, t)$ if it is in the right factor, we obtain a homotopy from $\mu \circ ((1 \times *) \vee (* \times 1))$ to the fold map $\nabla : S^n \vee S^n \to S^n$. Therefore, if S^n is an H–space, then

$$\partial D^{2n} \to S^n \vee S^n \xrightarrow{\nabla} S^n$$

is homotopic to

$$\partial D^{2n} \to S^n \vee S^n \xrightarrow{(1\times *)\vee(*\times 1)} S^n \times S^n \to S^n. \tag{1}$$

However, since

$$\partial D^{2n} \to S^n \vee S^n \xrightarrow{(1\times *)\vee(*\times 1)} S^n \times S^n$$

is equal to

$$\partial D^{2n} \to D^{2n} \to S^n \times S^n,$$

(1) extends to D^{2n}, and therefore is null-homotopic.

Conversely, suppose that

$$\partial D^{2n} \to S^n \vee S^n \xrightarrow{\nabla} S^n$$

is null-homotopic. Then it extends to a map $D^{2n} \to S^n$ making the following diagram commute

$$\begin{tikzcd}
\partial D^{2n} \arrow[r] \arrow[d] & S^n \vee S^n \arrow[r] \arrow[d] & S^n \\
D^{2n} \arrow[r] \arrow[urr] & S^n \times S^n \arrow[ur, dotted] &
\end{tikzcd}$$

Since the square is a pushout, we get a lift $S^n \times S^n \to S^n$. Further, by the commutativity of the right triangle, this gives S^n the structure of an H–space.

(4) Let X be 1–connected (i.e., $\pi_0 X = \pi_1 X = 0$). Recall that for a covering map $p : E \to B$ of based spaces, given a base point preserving map $f : X \to B$, there exists a unique base point preserving lift $\widetilde{f} : X \to E$ such that $p\widetilde{f} = f$.

(a) Prove that $\pi_n p : \pi_n E \to \pi_n B$ is an isomorphism for $n \geq 2$.

Solution. Note that for $n \geq 2$, S^n and $S^n \times I$ are both 1–connected. We need to prove that $\pi_n p$ is injective and surjective. Since any map $f : S^n \to B$ lifts uniquely to a map $f' : S^n \to E$ such that $p \circ f' = f$, $\pi_n p$ is surjective. Let $f, g : S^n \to E$ be such that $[p \circ f] = [p \circ g]$. Let $h : S^n \times I \to B$ be a homotopy between $p \circ f$ and $p \circ g$. Then there exists a unique lift $H : S^n \times I \to E$ such that $p \circ H = h$. In particular, $p \circ H|_0 = p \circ f$ and $p \circ H|_1 = p \circ g$. Therefore, since $H|_0$ lifts $p \circ f$ but so does f, we have $H|_0 = f$ and similarly, $H|_1 = g$. Therefore, $f \sim g$ and $[f] = [g]$ so that $\pi_n p$ is injective.

(b) Compute $\pi_k \mathbb{R}P^n$ in terms of $\pi_k S^n$. What about $\pi_k \mathbb{R}P^\infty$?

Solution. If $n = 1$, $\mathbb{R}P^1 \cong S^1$ so $\pi_k S^1 = \pi_k \mathbb{R}P^n$. Suppose that $n \geq 2$. Then let $p : S^n \to \mathbb{R}P^n$ be the antipodal map which is the quotient of S^n by the equivalence relation $-x \sim x$. This is a covering space. In fact, since S^n is simply connected, it is the universal cover of S^n. Since $|p^{-1}(*)| = 2$, $\pi_1 \mathbb{R}P^n$ is $\mathbb{Z}/2$. For $n \geq 2$, the previous problem implies that $\pi_k \mathbb{R}P^n \cong \pi_k S^n$.

Recall that $\mathbb{R}P^\infty = \cup_n \mathbb{R}P^n$. Define $S^\infty = \cup_{n=1}^\infty S^n$ with the union topology. Let $p : S^\infty \to \mathbb{R}P^\infty$ defined by the union of the map $p : S^n \to \mathbb{R}P^n$. Then $p : S^\infty \to \mathbb{R}P^\infty$ is a covering map. As before, $\pi_1 \mathbb{R}P^\infty \cong \mathbb{Z}/2$. Note that S^∞ is contractible, so $\pi_k \mathbb{R}P^\infty = 0$ for $n \geq 2$.