Introduction to the Steenrod Squares and the Hopf Invariant

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These notes are based on *Cohomology Operations and Applications in Homotopy Theory* by R. Mosher and M. Tangora

1 Cohomology Operations

Definition 1. *A cohomology operation of type $(\pi, n;\ G, m)$ is a family of functions $\theta_X : H^n(X;\pi) \to H^m(X;G)$, one for each space X, satisfying the naturality condition $f^*\theta_Y = \theta_X f^*$ for any map $f : X \to Y$.*

Remark 1. *Cohomology operations are natural transformations $H^n(-;\pi) \to H^m(-;G)$. By Brown's Representability theorem, this is equivalent to $[-, K(\pi, n)] \to [-, K(G, m)]$, which is equivalent to $[K(\pi, n), K(G, m)]$ using Yoneda's lemma. By applying Brown's representability Theorem again, we obtain the following theorem.*

Theorem 1. *There is a bijection*

$$\mathscr{O}(\pi, n;\ G, m) \leftrightarrow H^m(K(\pi, n); G),$$

given by $\theta \leftrightarrow \theta(\iota_n)$ where $\iota_n \in H^n(K(\pi, n); G); \pi)$ is the fundamental class of $K(\pi, n)$.

2 Steenrod Squares

Steenrod Squares are cohomology operations of type $(\mathbb{Z}_2, n;\ \mathbb{Z}_2, n+i)$.

Definition 2. *Let L be a subcomplex of the simplicial complex K. Steenrod Squares are group homomorphisms, $Sq^i : H^n(K, L; \mathbb{Z}_2) \to H^{n+i}(K, L; \mathbb{Z}_2)$, defined for $i \geq 0$, which have the following properties (which completely characterize the squaring operations, and may be taken as axioms)*

1. *Sq^i is a natural homomorphism $Sq^i : H^n(K, L; \mathbb{Z}_2) \to H^{n+i}(K, L; \mathbb{Z}_2)$.*
2. *If $i > n$, $Sq^i(x) = 0$ for all $x \in H^n(K, L; \mathbb{Z}_2)$.*
3. *$Sq^i(x) = x^2$ for all $x \in H^i(K, L; \mathbb{Z}_2)$.*
4. *Sq^0 is the identity homomorphism.*
5. *Sq^1 is the Bockstein homomorphism.*
6. *$\delta^* Sq^i = Sq^i \delta^*$ where $\delta^* : H^*(L; \mathbb{Z}_2) \to H^{*+1}(K, L; \mathbb{Z}_2)$.*
7. *Cartan formula: $Sq^i(xy) = \sum_j (Sq^j x)(Sq^{i-j} y)$.*
8. *Adem relations: For $a < 2b$, $Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$ where the binomial coefficient is taken mod 2.*

Remark 2. ***The Bockstein Homomorphism:***
The Bockstein homomorphism, $\beta : H^*(K, L; \mathbb{Z}_2) \to H^{*+1}(K, L; \mathbb{Z})$ arises from the exact coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$, however, this is not the Bockstein homomorphism referred to in the definition above. The composition of β with the reduction homomorphism gives a homomorphism

$$\beta \text{ mod } 2 = Sq^1 : H^n(K, L; \mathbb{Z}_2) \to H^{n+1}(K, L; \mathbb{Z}_2)$$

which is also called the Bockstein homomorphism, and is the homomorphism referred to in the definition of the Steenrod squares.

Definition 3. ***A Ring Homomorphism from the squares.***
Define $Sq : H^*(K;\mathbb{Z}_2) \to H^*(K;\mathbb{Z}_2)$ *by*

$$Sq(u) = \sum_i Sq^i(u)$$

Remark 3. *The sum is essentially finite since for* $i > dim(u)$ $Sq^i(u) = 0$, *and the image of* $Sq(u)$ *is in general not homogeneous.*

Proposition 1. *For* $u \in H^1(K;\mathbb{Z}_2), Sq^i(u^j) = \binom{j}{i} u^{j+i}$

Proof. Consider that $Sq(u) = Sq^0(u) + Sq^1(u) = u + u^2$, (properties (2)-(4)). Since Sq is a ring homomorphism, we have that

$$Sq(u^j) = (Sq(u))^j = (u + u^2)^j = u^j(1+u)^j = u^j \sum_k \binom{j}{k} u^k = \sum_i Sq^i(u^j),$$

and the proposition follows by comparing coefficients. □

Example: The Squares in action!
We note that $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[x]$. Conisder that from the previous proposition we have $Sq^i(x^j) = \binom{j}{i} x^{j+i}$. Thus,

$$Sq^i(x^j) = \begin{cases} x^{j+i} & \binom{j}{i} = 1 \bmod 2 \\ 0 & \binom{j}{i} = 0 \bmod 2 \end{cases}$$

3 The Hopf Invariant

Historical Context 1. *In the 30's Hopf studied homotopy classes of maps between spaces and started with a simple case: maps between spheres. In particular, he discovered an invariant on maps $S^{2n-1} \to S^{2n}$ which depended only on the homotopy class of the map. This construction was modernized as follows:*

Definition 4. *Given a map $f : S^{2n-1} \to S^n$, where $n > 1$, let K be the pushout given by f and the canonical inclusion*

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f"] \arrow[d] & S^n \arrow[d] \\ e^{2n} \arrow[r] & K \end{tikzcd}$$

Notice that $H^*(K;\mathbb{Z})$ has nonzero cohomology groups in degrees $0, n, 2n$ which are all isomorphic to $\mathbb{Z}$. Denote the generators by $1, \sigma, \tau$. The ***Hopf invariant*** *of f is the integer $H(f)$ defined by $\sigma^2 = H(f) \cdot \tau$.*

Remark 4. *Since the homotopy type of K depends on the homotopy class $[f]$, we have that $H(f)$ depends on $[f]$, hence is a homotopy invariant.*

Since we are traveling backwards in time, a natural question to ask is:

Question 1. *What does $H(f)$ look like as we vary n?*

The case where n is odd is extremely simple since we have anti-commutation of the cup product

$$\sigma^2 = -\sigma^2 \Rightarrow \sigma^2 = 0$$

hence $H(f)$ is always zero. If n is even, life becomes more complicated. A basic place to look is when K is the complex, quaternionic, or octonionic plane and one can find that these constructions give a map of Hopf invariant one. For reference, stare at the following pushout diagrams and their cohomology rings:

Case $n = 2$:

$$\begin{tikzcd} \mathbb{C}^2 \supset S^3 \arrow[r, "f"] \arrow[d] & S^2 \arrow[d] \\ e^4 \arrow[r] & \mathbb{C}P^2 \end{tikzcd}$$

Case $n = 4$:

$$\begin{tikzcd} \mathbb{H}^2 \supset S^7 \arrow[r, "f"] \arrow[d] & S^4 = \mathbb{H}P^1 \arrow[d] \\ e^8 \arrow[r] & \mathbb{H}P^2 \end{tikzcd}$$

Case $n = 8$:

$$\begin{tikzcd} \mathbb{O}^2 \supset S^{15} \arrow[r, "f"] \arrow[d] & S^8 = \mathbb{O}P^1 \arrow[d] \\ e^{16} \arrow[r] & \mathbb{O}P^2 \end{tikzcd}$$

$H^*(\mathbb{C}P^2) = \mathbb{Z}[H]/H^3$ the generator $H \rightsquigarrow \mathbb{C}P^1$ and $H^2 = 1 \cdot \tau$ where $\deg H = 2$
$H^*(\mathbb{H}P^2) = \mathbb{Z}[H]/H^3$ the generator $H \rightsquigarrow \mathbb{H}P^1$ and $H^2 = 1 \cdot \tau$ where $\deg H = 4$
$H^*(\mathbb{O}P^2) = \mathbb{Z}[H]/H^3$ the generator $H \rightsquigarrow \mathbb{O}P^1$ and $H^2 = 1 \cdot \tau$ where $\deg H = 8$
A couple natural questions to ask here are:

Question 2. *Does there exist any more maps with Hopf invariant one?*

Question 3. *Does the Hopf invariant contain any information about the homotopy groups of sphere?*

Both questions lead somewhere interesting but the first turns out to go very deep. Before tackling the second, consider the following

Proposition 2. *$H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a morphism of abelian groups.*

Proof. We prove this proposition by using the additive structure on homotopy groups and analyzing its construction in cohomology proving the homomorphism. So, consider the following pushout diagrams

$$\begin{tikzcd} S^{2n-1} \arrow[r, "Q"] \arrow[d] & S^{2n-1} \vee S^{2n-1} \arrow[r, "f \vee g"] & S^n \vee S^n \arrow[r, "\nabla"] \arrow[d] & S^n \arrow[ddr, bend left] & \\ e^{2n} \arrow[rr] \arrow[drrrr, bend right] & & (S^n \vee S^n) \sqcup_{(f \vee g)Q} e^{2n} = L \arrow[drr, dashed, "p"] & & \\ & & & & S^n \sqcup_{\nabla(f \vee g)Q} e^{2n} = K \end{tikzcd}$$

which gives a map p by universal properties. Now, if we consider the following two pushout diagrams

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f"] \arrow[d] & S^n \arrow[d] \\ e^{2n} \arrow[r] & S^n \sqcup_f e^{2n} \end{tikzcd}$$

$$\begin{tikzcd} S^{2n-1} \arrow[r, "g"] \arrow[d] & S^n \arrow[d] \\ e^{2n} \arrow[r] & S^n \sqcup_f e^{2n} \end{tikzcd}$$

we can use universal properties of coproducts to get the outer arrows from the inner pushout diagram

$$
\begin{tikzcd}
S^{2n-1} \arrow[r, "(f\vee g)Q"] \arrow[d] & S^n \vee S^n \arrow[d] \arrow[ddr, bend left] & \\
e^{2n} \arrow[r] \arrow[drr, bend right] & S^n \vee S^n \sqcup_{(f\vee g)Q} e^{2n} \arrow[dr, dashed, "q"] & \\
 & & (S^n \sqcup_f e^{2n}) \vee (S^n \sqcup_g e^{2n})
\end{tikzcd}
$$

giving us the desired map q.
Now, fixing generators

$$
\begin{aligned}
&\sigma \in H^n(K)\ \tau \in H^{2n}(K) \\
&\rho \in H^{2n}(L) \\
&\tau_1, \tau_2 \in H^{2n}(M) \\
&\sigma_1, \sigma_2 \in H^n(S^n \vee S^n) \text{ (also for } H^n(L))
\end{aligned}
$$

which are chosen such that

$$p^*\tau = q^*\tau_1 = q^*\tau_2 = \rho$$

and considering the following diagram in cohomology

$$H^*(K) \xrightarrow{p^*} H^*(L) \xleftarrow{q^*} H^*(M)$$

we have the following identities from the left arrow

$$
\begin{aligned}
&\sigma^2 = H(f+g)\tau \text{ by definition} \\
&p^*(\sigma^2) = (\sigma_1+\sigma_2)^2 \text{ and } p^*(H(f+g)\tau) = H(f+g)\rho
\end{aligned}
$$

and the identities from the right arrow

$$
\begin{aligned}
&(\sigma_1+\sigma_2)^2 = H(f)\tau_1 + H(g)\tau_2 \text{ by construction} \\
&q^*((\sigma_1+\sigma_2)^2) = (\sigma_1+\sigma_2)^2 \text{ and } q^*(H(f)\tau_1 + H(g)\tau_2) = H(f)\rho + H(g)\rho
\end{aligned}
$$

Then stringing these together gives the desired result. □

One can prove another technical proposition which is similar in flavor to previous, which states:

Proposition 3. *If n is even, then there exists a map of Hopf invariant 2.*

and since we get a non-zero morphism of an abelian group to $\mathbb{Z}$ we have the following

Corollary 1. *$\pi_{4n-1}(S^{2n})$ has direct summand $\mathbb{Z}$.*

Before going back to question (1), consider the following:

Definition 5. *We call a Steenrod square operation $Sq^i : H^n(X;\mathbb{Z}/2) \to H^{n+i}(X;\mathbb{Z}/2)$ indecomposable if it cannot we written as a sum of compositions of squaring operations*

$$\sum_{t<i} Sq^{I_t} \circ Sq^t$$

where $Sq^{I_t} = Sq^{i_k^t} \circ \cdots \circ Sq^{i_1^t}$.

One can prove the following theorem

Theorem 2. *Sq^i is indecomposable (it can not be written as $\sum_{t<i} a_t Sq^t$) if and only if $i = 2^k$.*

giving the key technical piece needed for proving a partial result about maps with Hopf invariant one:

Theorem 3. *If $f : S^{2n-1} \to S^n$ such that $H(f) = 1$, then $n = 2^k$.*

Proof. Since the Steenrod square of any degree k agrees with the squaring cup product, we have that

$$Sq^n(\sigma) = \sigma \smile \sigma = H(f) \cdot \tau = \tau$$

If Sq^i was decomposable, then we would have a sum

$$Sq^n(\sigma) = \sum_{t<i} Sq^{I_t} \circ Sq^t(\sigma) = \sum_{t<i} a_t \tau$$

for some non-zero integers a_t. But, all of the maps in the sum land in some degree k part of cohomology for $n < k < 2n$ which are zero, forcing $H(f) = 0$, a contradiction. □

I leave you with the following:

Question 4. *How does the Hopf Invariant relate to division algebras?*