Representability of Cohomology

1. Background

Goal: Edgar Brown proved the Brown representability theorem in 1962. Since then, the theorem has taken on more abstract and general forms. Our goal here will be to present a proof sketch of its original statement. We follow the proof given in *Algebraic Topology from a Homotopical Viewpoint* by Aguilar.

Theorem. (Brown representability theorem) Every Brown functor T is representable in the homotopy category of path-connected pointed CW complexes.

Yoneda lemma. Let T be a functor $T : \mathscr{C} \to \mathbf{Set}$. For any morphism $f : X \to C$ in $\mathscr{C}$, we have the morphism $Tf : T(C) \to T(X)$ in **Set**. Then any element $u \in T(C)$ for some object $C \in \mathscr{C}$ induces a natural transformation

$$\varphi_u : \mathrm{Hom}(-, C) \to T(-)$$

via $\varphi_u(f) := Tf(u)$.

Definition. A <u>universal element</u> of a functor $T : \mathscr{C} \to \mathbf{Set}$ is an element $u \in T(C)$ for some object $C \in \mathscr{C}$ such that u induces a natural isomorphism

$$\varphi_u : \mathrm{Hom}(-, C) \to T(-).$$

[This isomorphism is a sense in which we can say that T "works like Hom."]

Definition. Let $T : \mathscr{C} \to \mathbf{Set}$ be a Brown functor. An <u>n-universal element</u> is $u \in T(C)$ for some object $C \in \mathscr{C}$ such that the natural map

$$\varphi_u : \mathrm{Hom}(S^q, C) \to T(S^q)$$

is an isomorphism for $q < n$ and an epimorphism for $q = n$. The element is ∞-universal of it is n-universal for all $n \geq 1$.

2. Proof overview

- Take any space X and $v \in T(X)$. Construct the space $Y_1 \supset X$ and a 1-universal element $u_1 \in T(Y_1)$ such that $u_1|X = v$.
- Induct on n to get a space $Y_n \supset X$ and an n-universal element $u_n \in T(Y_n)$ such that $u_n|X = v$.
- Take the colimit of

  $$X \hookrightarrow Y_1 \hookrightarrow Y_2, \hookrightarrow \cdots$$

  to get Y and an ∞-universal element $u \in T(Y)$.
- The ∞-universal element <u>IS</u> the universal element we seek!

3. Constructing a 1-universal element

Take T to be a Brown functor, X a topological space and $v \in T(X)$.

Let $Y_1 = X \bigvee \left(\bigvee_{\alpha \in T(S^1)} S^1 \right)$. By the wedge axiom of Brown functors,

$$T(Y_1) = T(X) \times \prod_{\alpha \in T(S^1)} T(S^1).$$

Choose the element $u_1 \in T(Y_1)$ that is equivalent to $\big(v, (\alpha)_{\alpha \in T(S^1)}\big)$ under the equivalence given by the wedge axiom.

Claim: $\varphi_{u_1} : \mathrm{Hom}(S^1, Y_1) \to T(S^1)$ is surjective.
Proof: Take any $\alpha \in T(S^1)$. Let $i_\alpha : S^1 \to Y_1$ be the inclusion of S^1 into the α^{th} spot in the wedge. Using the wedge axiom and that i_α is an inclusion, it must be the case that Ti_α is the projection map onto the α^{th} coordinate. Then $\varphi_{u_1}(i_\alpha) = T_{i_\alpha}(u_1) = \alpha$ □

This proves that u_1 is a 1-universal element.

Lastly, we need to verify the claim that $u_1|X = v$. Let $i : X \to Y_1$ be the inclusion of X into the first coordinate of the wedge. Then $u_1|X = T_i(u_1) = v$.

4. **Constructing an n-universal element**

Assume inductively that we have constructed $Y_{n-1} \supset X$ with $(n-1)$-universal element $u_{n-1} \in T(Y_{n-1})$ such that $u_{n-1}|X = v$.

Let $Y'_n = Y_{n-1} \bigvee \left(\bigvee_{\alpha \in T(S^n)} S^n \right)$. In a similar fashion to the method described in the construction of the 1-universal element, we have the equivalence

$$T(Y'_n) = T(Y_{n-1}) \times \prod_{\alpha \in T(S^n)} T(S^n)$$

and we can let $u'_n \in T(Y'_n)$ corresponding to $\big(u_{n-1}, (\alpha)_{\alpha \in T(S^n)}\big)$. Then as before, $\varphi_{u'_n} : \mathrm{Hom}(S^n, Y'_n) \to T(S^n)$ is surjective.

We now wish to make $\varphi_{u'_n} : \mathrm{Hom}(S^{n-1}, Y'_n) \to T(S^{n-1})$ injective. To achieve this, consider $[f]$ in the kernel [1] of $\varphi_{u'_n}$. For each homotopy class $[f]$, attach an n-cell to Y'_n using $f : S^{n-1} \to Y'_n$ as the attaching map. Let this new space be called Y_n.

5. **Summary**

As it turns out, the space $Y = \mathrm{colim}\, Y_n$ and element $u \in T(Y)$ will give a representation of T. See Prop 12.2.21 in Aguilar for full details.

Notice that the axioms of reduced cohomology theories imply that cohomology is a Brown functor. Thus cohomology is representable. More specifically, cohomology is represented by a spectrum of Eilenberg-Maclane spaces.

[1] Note: $S^{n-1} = \Sigma S^{n-1}$ is an H-cospace, so $T(S^{n-1})$ has a group structure and the kernel of $\varphi_{u'_n}$ consists of elements that map to the group's identity element.