FROM SINGULAR HOMOLOGY TO SIMPLICIAL SETS

Any errors discovered in the reading should be forwarded to the author who will deal with them promptly.

1. Intuition

Fact 1.1. $0 \in \mathbf{N}$.

Throughout these notes, denote by CGHaus *the category of compactly generated Hausdorff spaces, and by* Top *we denote the category of spaces.*

Definition 1.2. Let $\mathbf{N}_- := \{-1\} \cup \mathbf{N}$. The set $\mathbf{N}_-$ is naturally an object of Ord, the category of partially ordered sets:

$$\{-1 \to 0 \to 1 \to 2 \to \cdots\}.$$

For each $n \in \mathbf{N}_-$ we define the **real n-simplex** by

$$|\triangle^n| := \begin{cases} \left\{ (x_0, \ldots, x_n) \in \mathbf{R}^{n+1}_{\geq 0} \,\middle|\, \sum_{i=0}^n x_i = 1 \right\} & n \geq 0 \\ \varnothing & n = -1 \end{cases}$$

topologized by the obvious inclusion $|\triangle^n| \subset \mathbf{R}^{n+1}$.

For most n, the sets

$$\mathrm{Hom}_{\mathsf{CGHaus}}\left(\left|\triangle^{n-1}\right|, |\triangle^n|\right)$$

and

$$\mathrm{Hom}_{\mathsf{CGHaus}}\left(\left|\triangle^{n+1}\right|, |\triangle^n|\right)$$

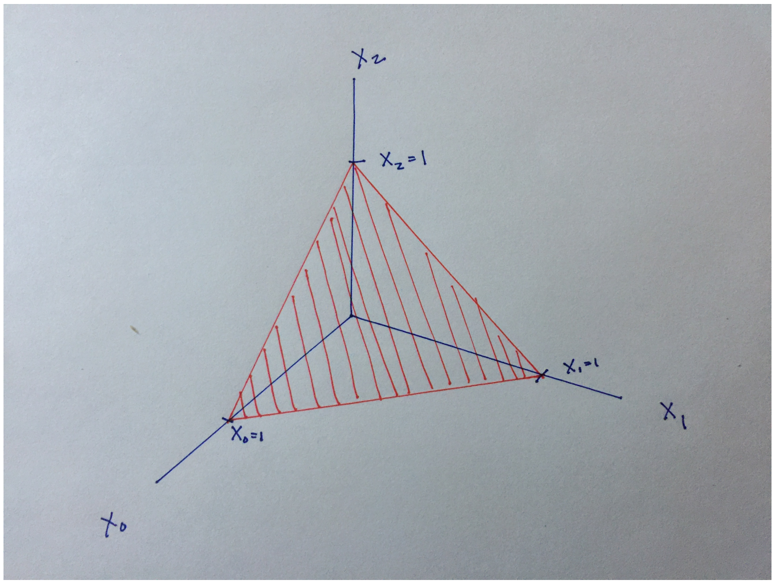

Figure 1. The real 2-simplex $|\triangle^2|$

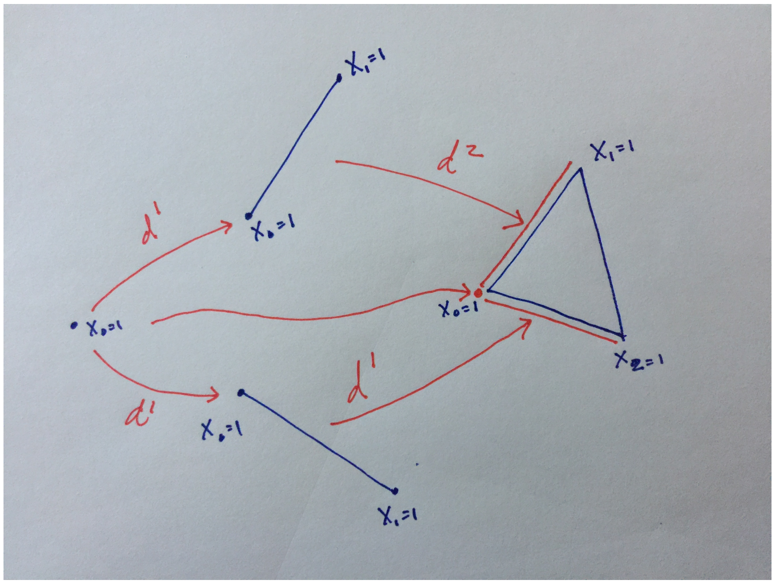

FIGURE 2. The commuting of d^i's.

are quite large. In particular, they've at least the cardinality of $|\triangle^n|$. There are however some morphisms in those Hom-sets which are easy to write down.

Definition 1.3. For $n \in \mathbf{N}_-$ and $0 \leq i \leq n$, we define morphisms of CGHaus, d^i and s^i, by the following rule.

$$\begin{tikzcd}
|\triangle^{n-1}| \arrow[r, "d^i"] & |\triangle^n| & |\triangle^{n+1}| \arrow[l, "s^i"'] \\
(x_0, x_1, \ldots, x_{n-1}) \arrow[r, mapsto] & (x_0, x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}) & \\
 & (x_0, x_1, \ldots, x_i + x_{i+1}, x_{i+2}, \ldots, x_n) & (x_0, x_1, \ldots, x_n) \arrow[l, mapsto]
\end{tikzcd}$$

Remark 1.4. We consider a pair of diagrams in CGHaus. Observe:

$$\begin{tikzcd}
 & |\triangle^1| \arrow[dr, "d^2"] & \\
|\triangle^0| \arrow[ur, "d^1"] \arrow[dr, "d^1"'] & & |\triangle^2| \\
 & |\triangle^1| \arrow[ur, "d^1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (1,0) \arrow[dr, mapsto] & \\
(1) \arrow[ur, mapsto] \arrow[dr, mapsto] & & (1,0,0)\,. \\
 & (1,0) \arrow[ur, mapsto] &
\end{tikzcd}$$

In a more geometric vein we envision this commutative square as in Figure 2.

Observe:

$$\begin{tikzcd}
 & |\triangle^0| \arrow[dr, "d^0"] & \\
|\triangle^1| \arrow[ur, "s^0"] \arrow[dr, "d^0"'] & & |\triangle^1| \\
 & |\triangle^2| \arrow[ur, "s^1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (1) \arrow[dr, mapsto] & \\
(x_0, x_1) \arrow[ur, mapsto] \arrow[dr, mapsto] & & (0,1)\,. \\
 & (0, x_0, x_1) \arrow[ur, mapsto] &
\end{tikzcd}$$

Again, in a more geometric vein, we envision this commutative square as in Figure 3.

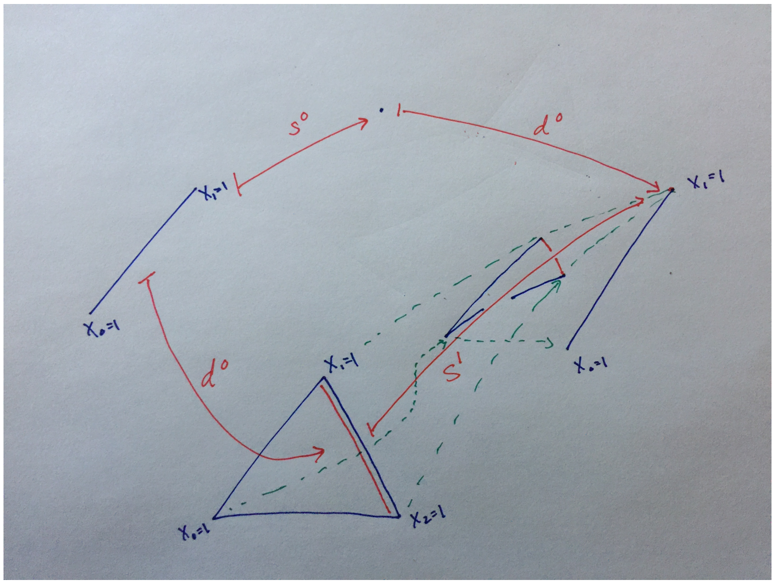

FIGURE 3. The commuting of d^i's and s^j's.

A trivial exercise:

Exercise 1.5. For each $n \in \mathbf{N}_-$, provide a cellular decomposition for the space $|\triangle^n|$, and do so in such manner that the maps s^j and d^i are cellular.

An exercise to be completed at least once in your life. You may also wait for the second instance of this exercise later on.

Exercise 1.6. Verify that for $n \in \mathbf{N}_-$ and i, j such that the compositions involved are defined, the morphisms d^i and s^j obey the following relations, called the **co-simplicial identities**.

$$\begin{cases} d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j \end{cases}$$

Definition 1.7. For each $n \geq 0$, and for each $0 \leq i \leq n$, let d_i, called a **face map**, to be the map

$$\begin{aligned} \mathrm{Hom}_{\mathsf{CW}}\left(|\triangle^n|, X\right) &\longrightarrow \mathrm{Hom}_{\mathsf{CW}}\left(|\triangle^{n-1}|, X\right) \\ f &\longmapsto f \circ d^i. \end{aligned}$$

We define

$$\mathsf{Sing}\,(X) := \left\{ \mathrm{Hom}\left(|\triangle^{-1}|, X\right) \xleftarrow{d_0} \mathrm{Hom}\left(|\triangle^{0}|, X\right) \underset{d_1}{\overset{d_0}{\leftleftarrows}} \mathrm{Hom}\left(|\triangle^{1}|, X\right) \Lleftarrow \cdots \right\},$$

and we consider this entity as a diagram in Set calling it **the singular set of** X. We define the object

$$\mathsf{S}_\bullet\left(\mathsf{X}\right) \in \mathsf{Ob}\left(\mathsf{AbGrp}^{(\mathbf{N}_-)^{\mathsf{op}}}\right),$$

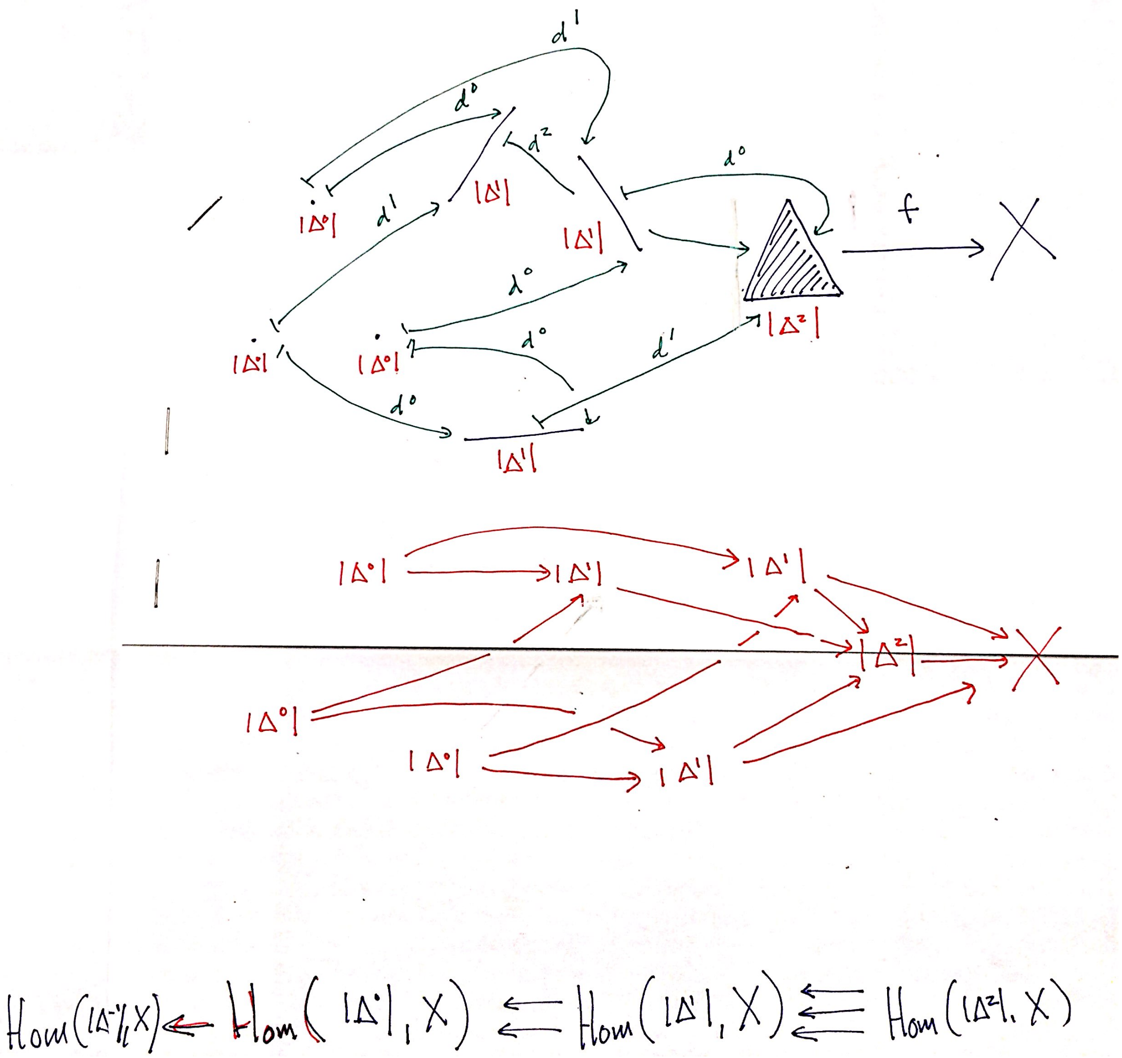

FIGURE 4. Fix $X \in \mathsf{Ob}(\mathsf{CGHaus})$, and fix $f : |\triangle^2| \longrightarrow X$ and consider the procession above, from a diagram in CGHaus to a diagram in Set.

by

$$\mathsf{S}_\bullet(X) := \cdots \leftarrow \mathbf{Z}\mathrm{Hom}\left(\left|\triangle^{n-1}\right|, X\right) \xleftarrow{\sum_{i=0}^{n}(-1)^{-1} d_i} \mathbf{Z}\mathrm{Hom}\left(\left|\triangle^{n}\right|, X\right) \leftarrow \cdots,$$

and refer to it as **the singular chain complex**.

Exercise 1.8. Two formal exercises:

- Prove the constructions $\mathsf{Sing}(X)$ and $\mathsf{S}_\bullet(X)$ to be functorial.
- Observe that the co-simplicial identities pass contravariantly to another set of identities, the simplicial identities, and as a consequence find that the singular chain complex is indeed a chain complex.

Proposition 1.9. *The composition*

$$\begin{tikzcd} \mathsf{CGHaus}_\bullet \arrow[rrr, "H_\bullet^{Sing}"] \arrow[dr, bend right, "forget"'] & & & \mathsf{Ch}\left(\mathsf{AbGrp}\right) \\ & \mathsf{CGHaus} \arrow[r, "\mathsf{S}_\bullet"'] & \mathsf{Ch}\left(\mathsf{AbGrp}\right) \arrow[ur, bend right, "H_\bullet"'] & \end{tikzcd}$$

is a reduced homology theory on $\mathsf{CGHaus}_\bullet$.

Proof. See Hatcher, May, or Goerss & Jardine. □

Were we to restrict ourselves to those spaces which are CW, then we've the following corollary.

Corollary 1.10. *We've a natural isomorphism of chains of abelian groups,*

$$H_\bullet^{Sing}\left(X\right) \xrightarrow{\sim} H_\bullet^{\mathsf{CW}}\left(X\right).$$

What about the data we had assembled right before $\mathsf{Sing}\left(X\right)$, what kind of thing was it?

- a diagram in CGHaus or equivalently a subcategory of CGHaus; or
- a diagram in ${}^{\mathsf{CGHaus}}/_{X}$ or equivalently a subcategory of ${}^{\mathsf{CGHaus}}/_{X}$.

Definition 1.11. Let $|\triangle| \downarrow X$ be the category such that:

$$\mathsf{Ob}\left(|\triangle| \downarrow X\right) = \left\{ |\triangle^n| \to X : \mathsf{CGHaus} \,\middle|\, n \in \mathbf{N}_- \right\}$$

and

$$\mathrm{Hom}_{|\triangle|\downarrow X}\left(\begin{array}{c} |\triangle^n| \\ \downarrow \\ X \end{array}, \begin{array}{c} |\triangle^m| \\ \downarrow \\ X \end{array} \right) = \left\{ \begin{tikzcd} |\triangle^n| \arrow[rr, "\theta"] \arrow[dr] & & |\triangle^m| \arrow[dl] \\ & X & \end{tikzcd} \,\middle|\, \theta = d^{i_k}\cdots d^{i_1} s^{j_\ell}\cdots s^{j_1} \right\}.$$

Let $F : |\triangle| \downarrow X \longrightarrow \mathsf{CGHaus}$ be the forgetful functor $\left(|\triangle^n| \to X\right) \longmapsto |\triangle^n|$.

Remark 1.12. Observe that $F\left(|\triangle| \downarrow X\right)$ is a diagram in CGHaus, and more, since for each $|\triangle^n|$ appearing in that diagram we've maps $|\triangle^n| \to X$ which commute with the morphisms of the diagram, we get a map

$$\varinjlim_{F(|\triangle|\downarrow X)} |\triangle^n| \longrightarrow X.$$

It must also be noted that if X is pointed, then this colimit is naturally pointed by the 0-simplex corresponding to the point $|\triangle^0| = \bullet \longrightarrow X$.

Lemma 1.13. The induced map

$$\varinjlim_{F(|\triangle|\downarrow X)} |\triangle^n| \longrightarrow X$$

is a weak homotopy equivalence.

Proof. It suffices to prove that for each $n \in \mathbf{N}$, and all $X \in \mathsf{Ob}\left(\mathsf{CGHaus}\right)$ that the induced morphism of groups

$$\pi_n\left(\varinjlim_{F(|\triangle|\downarrow X)} |\triangle^n| \right) \longrightarrow \pi_n\left(X\right)$$

is an isomorphism, and for that lesser criterion a proof of bijectivity will suffice.

Injectivity: suppose

$$S^n \longrightarrow \varinjlim_{F(|\triangle|\downarrow X)} |\triangle^n|$$

to generate a homotopy class of spheres in that colimit which passes to the trivial class in X. We may thus assemble the following commutative diagram.

$$\begin{tikzcd}
S^n \arrow[rrr] \arrow[ddd] & & & \varinjlim_{F(|\triangle|\downarrow X)} |\triangle^\bullet| \arrow[ddd] \\
& & |\triangle^{n+1}| \arrow[ur, dashed] \arrow[ddr] & \\
& D^{n+1} \arrow[ur, "\sim"] \arrow[drr] & & \\
S^n \wedge I_+ \arrow[ur, "\sim"] \arrow[rrr, "\text{nullhomotopy}"'] & & & X
\end{tikzcd}$$

The first isomorphism is from a common characterization of a nullhomotopy, the second isomorphism is evident; the indicated lift is the canonical inclusion of $|\triangle^{n+1}|$ into the colimit corresponding to the morphism $|\triangle^{n+1}| \longrightarrow X$. Then since $|\triangle^{n+1}|$ is contractible, so too the image of S^n in the colimit. Indeed $[S^n \longrightarrow X] = [\bullet]$ and the induced morphism of homotopy groups is injective.

Surjectivity: suppose $S^n \longrightarrow X$ to be some morphism of CGHaus. Then we may assemble the following diagram.

$$\begin{tikzcd}
& & & \varinjlim_{F(|\triangle|\downarrow X)} |\triangle^\bullet| \arrow[ddd] \\
& & \partial\,|\triangle^{n+1}| \arrow[ur, dashed] \arrow[ddr] & \\
& \partial D^{n+1} \arrow[ur, "\sim"] \arrow[drr] & & \\
S^n \arrow[ur, "\sim"] \arrow[rrr] & & & X
\end{tikzcd}$$

In this diagram the isomorphisms are evident and the lift is gotten by the universal property of the colimit on the sub-diagram of $F(|\triangle| \downarrow X)$ corresponding to the $n+2$ faces $\partial\,|\triangle^{n+1}|$ in X. Thus the induced morphism of homotopy groups is surjective. □

2. Formalism

We have already observed that $\mathsf{S}_\bullet(X)$ may be thought of either as a diagram in AbGrp or as a functor $(\mathbf{N}_-)^{\mathsf{op}} \longrightarrow \mathsf{AbGrp}$, i.e. an object of the functor category $\mathsf{AbGrp}^{(\mathbf{N}_-)^{\mathsf{op}}}$. Until this point we've thought of $\mathsf{Sing}(X)$ only as a diagram in Set, but as with $\mathsf{S}_\bullet$ we may envision this diagram as a functor, with as of yet unnamed source.

Definition 2.1. Let $\triangle_-$, called the **augmented simplex category**, be the full subcategory of Ord subtended by the objects

$$\mathsf{Ob}\left(\triangle_-\right) := \left\{ [n] := \begin{cases} \{0 < 1 < \cdots < n\} & n \geq 0 \\ \varnothing & n = -1 \end{cases} \right\}.$$

Lemma 2.2. *The category $\triangle_-$ is generated by two types of morphisms:*

$$\begin{tikzcd}
{[n-1]} \arrow[r, "d^i"] & {[n]} & {[n+1]} \arrow[l, "s^i"'] \\
0 \arrow[r, mapsto] & 0 & 0 \arrow[l, mapsto] \\
1 \arrow[r, mapsto] & 1 & 1 \arrow[l, mapsto] \\
\vdots & \vdots & \vdots \\
i-1 \arrow[r, mapsto] & i-1 & i-1 \arrow[l, mapsto] \\
i \arrow[dr, mapsto] & i & i \arrow[l, mapsto] \\
\vdots & i+1 & i+1 \arrow[ul, mapsto] \\
n-1 \arrow[ddr, mapsto] & \vdots & i+2 \arrow[ul, mapsto] \\
 & & \vdots \\
 & n & n+1 \arrow[l, mapsto]
\end{tikzcd}$$

and these morphism satisfy the co-simplicial identities.

Proof. Exercise. □

Definition 2.3. There are two important variations on $\triangle_-$. Let $\triangle$ be the full subcategory of $\triangle_-$ subtended by the objects $[n]$ for $n \in \mathbf{N}$, and call this the **simplex category**. Denote by $\triangle^+$the wide subcategory of $\triangle$ wherein morphisms are generated only by the d^i, and call it the **semi-simplex category**. Denote likewise by $(\triangle_-)^+$ the wide subcategory of $\triangle_-$ wherein morphisms are generated by only by the d^i, and call this the **augmented semi-simplex category**.

A functor

$$(\triangle_-)^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

is called an **augmented simplicial set**. A functor

$$\triangle^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

is called a **simplicial set**. A functor

$$\left(\triangle^+\right)^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

is called a **semi-simplicial set**. Lastly, a functor

$$\left(\triangle_-^+\right)^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

is called an **augmented semi-simplicial set**.

Remark 2.4. The profusion of categories here may seem unnecessary, but there are certain advantages. Our implicit use of $\triangle_-$ until now, as oppose to $\triangle$ is precisely why we get a reduced homology theory, as opposed to a homology theory. Similarly, it will be seen that generalization of the construction $\varinjlim_{F(|\triangle|\downarrow X)} |\triangle^\bullet|$, did not depend on keeping track of those maps s^j between simplices, so the categories $\triangle_-^+$and $\triangle^+$ will do just as well for that construction in some ways. This is not to say that the maps s^j are irrelevant in general, in fact they are essential for products as will be seen later.

We've already seen one instance of such an object, $\mathsf{Sing}\,(X)$, but there are of course others. Perhaps the most important for our purposes amongst these simplicial sets are the representable pre-sheaves.

$$\triangle^n := \mathrm{Hom}\,(\_\,, [n])$$

Exercise 2.5. Prove that the assignment

$$\begin{aligned} |\_| : \triangle &\longrightarrow \mathsf{CGHaus} \\ \triangle^n &\longmapsto |\triangle^n| \end{aligned}$$

is functorial.

We constructed the pre-sheaves $\mathsf{Sing}\,(X)$ from diagrams in CGHaus, and what's more, we proved that the colimits over those diagrams are weakly equivalent to the original spaces X. We will now discover how, in fact, every simplicial set corresponds to a diagram in a similar way, and how those diagrams can be used to beget spaces in such a way that we recover the construction performed with $\mathsf{Sing}\,(X)$.

Towards the extraction from an arbitrary pre-sheaf the correct diagram the following famous lemma suggests the way.

Lemma 2.6 (the Yoneda Lemma)**.** *Let $\mathscr{C}$ be a small category. Letting, for each $X \in \mathsf{Ob}\,(\mathscr{C})$, $h_X := Hom_{\mathscr{C}}\,(\_\,, X)$, then for each $Y \in \mathsf{Ob}\left(\mathsf{Set}^{\mathscr{C}^{\mathsf{op}}}\right)$, we've an isomorphism of sets*

$$Hom_{\widehat{\mathscr{C}}}\,(h_X, Y) \xrightarrow{\sim} Y\,(X)$$

which is moreover natural in both X and Y.

Corollary 2.7 (the Yoneda Embedding)**.** *The functor $\mathscr{C} \longrightarrow \mathsf{Set}^{\mathscr{C}^{\mathsf{op}}} : X \longmapsto h_X$ is a full and faithful embedding.*

Definition 2.8. Given a pre-sheaf $X : \triangle^{\mathsf{op}} \longrightarrow \mathsf{Set}$, define the category $\triangle \downarrow X$ to be the category given by setting

$$\mathsf{Ob}\,(\triangle \downarrow X) := \left\{ \triangle^n \longrightarrow X : \mathsf{Set}^{\triangle^{\mathsf{op}}} \middle|\, n \in \mathbf{N} \right\},$$

and

$$\mathrm{Hom}_{\triangle\downarrow X}\left(\begin{array}{c} \triangle^n \\ \downarrow \\ X \end{array}, \begin{array}{c} \triangle^m \\ \downarrow \\ X \end{array} \right) := \left\{ \begin{tikzcd} \triangle^n \arrow[rr, "\theta"] \arrow[dr] & & \triangle^m \arrow[dl] \\ & X & \end{tikzcd} \right\},$$

with the obvious composition law.

Exercise 2.9. Prove the assignment

$$\begin{aligned} \mathsf{Set}^{\triangle^{\mathsf{op}}} &\longrightarrow \mathsf{Cat} \\ X &\longmapsto \triangle \downarrow X \end{aligned}$$

to be functorial.

Just as we did earlier, we may choose to let F be the obvious functor $(\triangle^n \to X) \longmapsto \triangle^n$, whence $F(\triangle \downarrow X)$ is a diagram in $\mathsf{Set}^{\triangle^{\mathsf{op}}}$. What's more however, it is not merely a diagram in $\mathsf{Set}^{\triangle^{\mathsf{op}}}$ but in fact a diagram therein contained in the image of $\triangle$ under the Yoneda embedding.

Corollary 2.10. *For any $X \in \mathsf{Ob}\left(\mathsf{Set}^{\triangle^{\mathsf{op}}}\right)$ we've an isomorphism $\varinjlim_{F(\triangle\downarrow X)} \triangle^\bullet \xrightarrow{\sim} X$ which is natural in X.*

Definition 2.11. We define the **geometric realization** of a simplicial set by

$$\begin{aligned} |\_| : \widehat{\triangle} &\longrightarrow \mathsf{CGHaus} \\ X &\longmapsto |X| := \varinjlim_{F(\triangle\downarrow X)} |\triangle^\bullet| . \end{aligned}$$

where the colimit $\varinjlim_{F(\triangle\downarrow X)} |\triangle^\bullet|$ is the colimit over the diagram in CGHaus gotten by applying $|\_| : \triangle^n \longmapsto |\triangle^n|$ to the diagram $F(\triangle \downarrow X)$ in $\mathsf{Set}^{\triangle^{\mathsf{op}}}$.

Remark 2.12. We use the same notation as we did for $\triangle^n \longmapsto |\triangle^n|$ as this new functor *extends* the prior use in a way which admits formalization[1]. More, it must be noted that the way in which we extend this functor preserves colimits. In particular note that for any $X \in \mathsf{Ob}\left(\mathsf{Set}^{\triangle^{\mathsf{op}}}\right)$ we've

$$\left| \varinjlim_{F(\triangle\downarrow X)} \triangle^\bullet \right| \xleftarrow{\sim} |X| = \varinjlim_{F(\triangle\downarrow X)} |\triangle^\bullet| .$$

Exercise 2.13. Suppose $\mathcal{I}$ to be a small category and suppose

$$\begin{aligned} \mathcal{I} &\longrightarrow \mathsf{Top} \\ i &\longmapsto X_i \end{aligned}$$

to be a diagram in Top. Prove that the space

$$\textstyle\coprod_{i \in \mathsf{Ob}(\mathcal{I})} X_i \big/ \sim$$

where $X_i \ni x \sim y \in X_j$ if and only if there exists a chain

$$\begin{tikzcd}
X_i & & X_1 & & X_j \\
x & X_0 \arrow[ul] \arrow[ur] & x_1 & \cdots \arrow[ul] \arrow[ur] & y \\
& x_0 \arrow[ul, mapsto] \arrow[ur, mapsto] & & \cdots \arrow[ul, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}$$

of objects $X_0, X_1, \ldots$, morphisms in the diagram, and elements as above, enjoys the universal property of $\varinjlim X_i$.

[1] In fact it is a Kan extension along the Yoneda embedding.

Remark 2.14. You should convince yourself that $|\mathsf{Sing}\,(X)|$ got this way is the same space as $\varinjlim_{|\triangle|\downarrow X} |\triangle^\bullet|$ which we've already proved weakly equivalent to X.

Using what you proved in the previous exercise, observe that for a simplicial set X, the space $|X|$ is isomorphic to the space

$$\amalg_{X(0)}|\triangle^0|\amalg_{X(1)}|\triangle^1|\amalg_{X(2)}|\triangle^2|\cdots/\sim$$

where the relation $\sim$ is generated by

$$\left|\triangle^{n-1}\right| \ni x \sim d^i\,(x) \in |\triangle^n|\,.$$

Exercise 2.15. Convince yourself that what we've proven in this exercise is enough to verify that geometric realization is concerned only with the morphisms coming from $\triangle^+$ or $\triangle^+_-$, i.e. prove that given any $X \in \mathsf{Ob}\left(\mathsf{Set}^{\triangle^{\mathrm{op}}_-}\right)$, that

$$\varinjlim_{F(\triangle^+\downarrow X)} |\triangle^\bullet| \overset{\sim}{\to} \varinjlim_{F\left(\triangle^+_-\downarrow X\right)} |\triangle^\bullet| \overset{\sim}{\to} \varinjlim_{F(\triangle\downarrow X)} |\triangle^\bullet| \overset{\sim}{\to} \varinjlim_{F(\triangle_-\downarrow X)} |\triangle^\bullet|\,.$$

Remark 2.16. Degeneracies however are extremely important for agreement of our geometric notion of the product on spaces, with the categorical notion on simplicial sets. See homework 4 problem 4.

As is suggested by our development of them in parallel, there is a formal relationship between the functors $|\_|$ and Sing.

Proposition 2.17. *The functors*

$$\begin{tikzcd} \mathsf{Set}^{\triangle^{\mathrm{op}}} \arrow[rr, bend right, "|\_|"'] \arrow[rr, phantom, "\perp"] & & \mathsf{CGHaus} \arrow[ll, bend right, "\mathsf{Sing}"'] \end{tikzcd}\,.$$

comprise an adjunction.

Indeed, the proof of the adjunction will now seem trivial. In fact, the motivation for the tack we have taken through this material was precisely to make this so [2].

[2]in all truthfulness, it has also been to suggest a way that the Yoneda lemma, all too often seen merely as abstract formalism, admits a natural geometric interpretation.

Proof. Let $X \in \mathsf{Ob}\left(\mathsf{Set}^{\triangle^{\mathsf{op}}}\right)$ and let $Y \in \mathsf{Ob}\left(\mathsf{CGHaus}\right)$. Then observe that

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{CGHaus}}\left(\left|X\right|, Y\right) &\tilde{\to} \mathrm{Hom}_{\mathsf{CGHaus}}\left(\varinjlim_{F(\triangle\downarrow X)} \left|\triangle^{\bullet}\right|, Y\right) \\
&\tilde{\to} \varprojlim_{F(\triangle\downarrow X)} \mathrm{Hom}_{\mathsf{CGHaus}}\left(\left|\triangle^{\bullet}\right|, Y\right) \\
&= \varprojlim_{F(\triangle\downarrow X)} \mathrm{Hom}_{\mathsf{CGHaus}}\left(\triangle^{\bullet}, \mathsf{Sing}\left(Y\right)\right) \\
&\tilde{\to} \mathrm{Hom}_{\mathsf{Set}^{\triangle^{\mathsf{op}}}}\left(\varinjlim_{F(\triangle\downarrow X)} \triangle^{\bullet}, \mathsf{Sing}\left(Y\right)\right)
\end{aligned}$$

where the first isomorphism was observed after the definition of the geometric realization, the second is the universal property, the equality is by definition, and the last isomorphism is again by universal property. $\square$