MATH 6280 - NOTES ON THE HOMOLOGY AND COHOMOLOGY OF $\mathbb{R}P^n$

We will study the antipodal map $a_n : S^n \to S^n$ which sends $\mathbf{x} \in S^n$ to $-\mathbf{x}$.

Remark 0.1. Note that the antipodal map is not base point preserving. To make this completely precise, we have to define the degree of an unbased map. One way to do this is to say that any map is homotopic to a cellular map. For S^n, this implies that every element of $[S^n, S^n]$ is homotopic to one in $[S^n, S^n]_*$ and use this to define the degree.

Let ΣS^{n-1} be the un-reduced suspension

$$\Sigma S^{n-1} = \{(\mathbf{y}, t) \mid \mathbf{y} \in S^{n-1}, 0 \leq t \leq 1\}/((\mathbf{y}, 1) \sim (\mathbf{y}', 1), (\mathbf{y}, 0) \sim (\mathbf{y}', 0))$$

We will use the following identification:

$$i_n : S^n \to \Sigma S^{n-1}$$

which, for $\mathbf{x} = (x_1, \ldots, x_n)$, sends

$$i_n(\mathbf{x}, x_{n+1}) = \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, \frac{1 + x_{n+1}}{2} \right).$$

Note then that the antipodal map is

$$a_n(\mathbf{y}, t) = (-\mathbf{y}, 1 - t).$$

Lemma 0.2. *The degree of the antipodal map $a_n : S^n \to S^n$ is $(-1)^{n+1}$.*

Proof. That a_1 is homotopic to the identity is clear by composition it with a rotation of the plane. Suppose that a_{n-1} has degree $(-1)^n$. Recall that for $\Sigma f(x, t) = (f(x), t)$, we have $-\Sigma f(x, t) = (f(x), 1 - t)$. Then note that

$$a_n(\mathbf{y}, t) = (-\mathbf{y}, 1 - t) = -\Sigma a_{n-1}.$$

Since a_{n-1} has degree $(-1)^n$ and $\Sigma : \pi_{n-1} S^{n-1} \to \pi_n S^n$ is an isomorphism taking the identity to the identity, then Σa_{n-1} has degree $(-1)^n$. So, the claim follows from the fact that $\deg(-\Sigma f) = -\deg(f)$. Indeed, $-\Sigma f(x) = f(x) \wedge (1 - t)$ was how we defined the group inverse! $\square$

Like in *Concise*, we start by giving a different cell structure to S^n. We build S^n inductively so that:

- $(S^n)^q = S^q$, where S^q is the subspace of S^n whose last $n - q$–coordinates are zero.

- S^n has two q-cell for each $0 \leq q \leq n$, namely e_+^q which are the points of S^q such that the last coordinate is greater or equal to zero, and e_-^q, the points of S^q such that the last coordinate is less than or equal to zero.

We have:

$$e_+^q \cup e_-^q = S^q \quad \text{and} \quad e_+^q \cap e_-^q = S^{q-1}.$$

Fix homeomorphisms for the cells as

$$\psi_+^q : D^q \to S^q \qquad\qquad \psi_+^q(x_1, \dots, x_q) = (x_1, \dots, x_q, (1 - \sum x_i^2)^{1/2})$$

and

$$\psi_-^q : D^q \to S^q \qquad\qquad \psi_-^q(x_1, \dots, x_q) = (-x_1, \dots, -x_q, -(1 - \sum x_i^2)^{1/2})$$

So, we have:

$$\begin{tikzcd}
S^{q-1} \vee S^{q-1} \arrow[r, "\phi_+^q \vee \phi_-^q"] \arrow[d] & S^{q-1} \arrow[d] \\
D^q \vee D^q \arrow[r, "\psi_+^q \vee \psi_{-1}^q"] & S^q.
\end{tikzcd}$$

In particular,

$$\phi_+^q = \mathrm{id} \quad \text{and} \quad \phi_-^q = a_{q-1}$$

Further, let

$$\pi_+^q : S^q \to e_+^q / S^{q-1} \quad \text{and} \quad \pi_-^q : S^q \to e_-^q / S^{q-1}$$

Choose

$$i_{q-1} : D^{q-1}/S^{q-2} \to S^{q-1}$$

so that the composite $i_{q-1} \circ (\psi_+^{q-1})^{-1} \circ \pi_+^{q-1}$ has degree one. We have a commutative diagram:

$$\begin{tikzcd}
S^{q-1} \arrow[dr, "\phi_-^q"] & & & & \\
& S^{q-1} \arrow[r, "\pi_+^{q-1}"] \arrow[d, "a_{q-1}"'] \arrow[rrr, bend left, "\deg 1"] & e_+^{q-1}/S^{q-2} \arrow[d, "a_{q-1}"'] & D^{q-1}/S^{q-2} \arrow[l, "\psi_+^{q-1}"'] \arrow[d, equal] \arrow[dr, "i_{q-1}"] & \\
S^{q-1} \arrow[ur, "\phi_+^q"] \arrow[dr, "\phi_-^q"'] & & & & S^{q-1} \\
& S^{q-1} \arrow[r, "\pi_-^{q-1}"'] & e_-^{q-1}/S^{q-2} & D^{q-1}/S^{q-2} \arrow[l, "\psi_-^{q-1}"] \arrow[ur, "i_{q-1}"'] & \\
S^{q-1} \arrow[ur, "\phi_+^q"'] & & & &
\end{tikzcd}$$

So we can use the diagram and the fact that a_{q-1} has degree $(-1)^q$ compute the various degrees:

$$d_q[\psi_+^q] = [\psi_+^{q-1}] + (-1)^q[\psi_-^{q-1}]$$
$$d_q[\psi_-^q] = (-1)^q[\psi_+^{q-1}] + [\psi_-^{q-1}].$$

This gives all the information we need to compute $H_*(C_*(S^n))$ (actually, this is also works for $C_*(S^\infty)$).

Now, give $\mathbb{R}P^n$ one cell $\psi^q : D^q \to \mathbb{R}P^q$ in each degree. The double cover is a cellular map for these cell structures and

$$C_*(\mathbb{R}P^n) \cong C_*(S^n)/([\psi_+^q] = [\psi_-^q]).$$

Further, $d_q : C_q(\mathbb{R}P^n) \to C_{q-1}(\mathbb{R}P^n)$ is

$$d_q[\psi^q] = [\psi^{q-1}] + (-1)^q[\psi^{q-1}].$$

So, the cellular chain complex is

$$0 \to C_m(\mathbb{R}P^m) \xrightarrow{1+(-1)^m} C_{m-1}(\mathbb{R}P^m) \xrightarrow{1+(-1)^{m-1}} \ldots \xrightarrow{0} C_2(\mathbb{R}P^m) \xrightarrow{2} C_1(\mathbb{R}P^m) \xrightarrow{0} C_0(\mathbb{R}P^m) \to 0$$

So

$$H_n(\mathbb{R}P^m) = \begin{cases} \mathbb{Z} & n = 0 \text{ or } n = m \text{ and } m \text{ is odd} \\ \mathbb{Z}/2 & n < m \text{ and } n \text{ is odd.} \end{cases}$$

The cellular cochain complex is

$$0 \leftarrow C^m(\mathbb{R}P^m) \xleftarrow{1+(-1)^m} C^{m-1}(\mathbb{R}P^m) \xleftarrow{1+(-1)^{m-1}} \ldots \xleftarrow{0} C^2(\mathbb{R}P^m) \xleftarrow{2} C^1(\mathbb{R}P^m) \xleftarrow{0} C^0(\mathbb{R}P^m) \leftarrow 0$$

So

$$H^n(\mathbb{R}P^m) = \begin{cases} \mathbb{Z} & n = 0 \text{ and } n = m \text{ if } m \text{ is odd} \\ \mathbb{Z}/2 & 0 < n \leq m \text{ and } n \text{ is even.} \end{cases}$$