Topological K-theory

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The idea of topological K-theory is that spaces can be distinguished by the vector bundles they support. Below we present the basic ideas and definitions (vector bundles, classifying spaces) and state one major theorem (Bott periodicity), finally presenting K-theory in the context of a generalized cohomology theory via its representability by an Ω-prespectrum. We then indicate an application (the only spheres that are H-spaces are S^k for $k = 0, 1, 3, 7$) and other directions that K-theory can take (algebraic K-theory and the K-theory of C^* algebras).

1 Vector Bundles and $K(X)$

Definition 1. *A* **vector bundle** *(of dimension n over $k = \mathbb{R}$ or $\mathbb{C}$) with base space B and total space E is a map $E \xrightarrow{p} B$ with the following properties.*

- *E locally a product: there is a an open cover $\cup_i U_i$ of B and homeomorphisms*

$$\phi : p^{-1}(U_i) \xrightarrow{\cong} U_i \times k^n$$

  such that $(p \circ \phi_i^{-1})(u, v) = u$.

- *Transitions are linear: the homeomorphisms*

$$\phi_j \circ \phi_i^{-1}|_{(U_i \cap U_j)\times k^n} : (U_i \cap U_j) \times k^n \to (U_i \cap U_j) \times k^n.$$

  are linear isomorphisms in each fiber $\{b\} \times k^n$.

One can define vector bundles over X via **clutching functions** as follows. Let $X = \cup_i U_i$ be an open cover and $E_i = U_i \times \mathbb{C}^n$ a trivial bundle over each element of the cover. Suppose we have $g_{ij} : U_i \cap U_j \to GL_n(\mathbb{C})$ satisfying the cocycle condition $g_{jk} \circ g_{ij} = g_{ik}$ on $U_i \cap U_j \cap U_k$. Then $\coprod_i E_i / \sim$ is a vector bundle over X (projecting onto the first coordinate), where

$$(x, v) \sim (x, g_{ij} v), \ x \in U_i \cap U_j.$$

Some examples:

- the trivial bundles $E = B \times k^n$,
- tangent bundles of smooth manifolds and normal bundles of immersed manifolds,
- the canonical bundle over $P^1(\mathbb{C})$ (lines through the origin in $\mathbb{C}^2$),

$$H = \{(L, v) : L \in P^1(\mathbb{C}), v \in L\}.$$

- the Möbius bundle $[0,1]\times\mathbb{R}/\sim$, where we identify $(0,x)$ and $(1,-x)$ (this is the canonical bundle over $S^1 \cong P^1(\mathbb{R})$).

The canonical bundle over $P^1(\mathbb{C})$ will be important later.

There are additive and multiplicative operations on vector bundles over a given base. Given two k-vector bundles over the same base, $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B$ (say of dimensions n and m respectively), we have

$$\begin{cases} E \oplus E' \xrightarrow{p \oplus p'} B \\ (p \oplus p')^{-1}(b) = p^{-1}(b) \oplus p'^{-1}(b) \end{cases}, \quad \begin{cases} E \otimes E' \xrightarrow{p \otimes p'} B \\ (p \otimes p')^{-1}(b) = p^{-1}(b) \otimes p'^{-1}(b) \end{cases},$$

the fibrewise sum and product of the vector bundles (of dimensions $n+m$ and nm respectively). If $B = \cup_i U_i$ is such that E, E' are trivial over U_i and $\{g_{ij}\}$, $\{g'_{ij}\}$ are their respective clutching functions, then the clutching functions for $E \oplus E'$ and $E \otimes E'$ are

$$\{g_{ij} \oplus g'_{ij} \in GL_{n+m}(k)\},\ \{g_{ij} \otimes g'_{ij} \in GL_{nm}(k)\}.$$

These operation give the isomorphism classes of k-vector bundles over B, $\mathrm{Vect}_k(B)$, the structure of a commutative semi-ring. We can formally extend to *virtual bundles*, similar to the construction of $\mathbb{Z}$ from $\mathbb{N}$.

Proposition 1. *Given a semi-ring A there is a ring R and homomorphism $A \to R$ such that for any homorphism $A \to R'$ from A to a ring R', there is a unique homorphism $R \to R'$ such that*

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & R' \\ & R \arrow[u] \end{tikzcd}$$

commutes. Furthermore, this constructin is functorial in A. [The abelian group $G(A)$ obtained from an abelian semi-group A in the constuction below is called the ***Grothendieck group*** *of A.]*

Proof. Let R be the set $A \times A$ modulo the equivalence relation $(a,b) \sim (a',b')$ if there exists $c \in A$ such that

$$a + b' + c = a' + b + c.$$

Addition and multiplication are defined by

$$(a,b) + (c,d) = (a+b, c+d),\ (a,b)\cdot(c,d) = (ac+bd, ad+bc).$$

Another construction is to take the free abelian group on A and quotient by the appropriate relations

$$\{(a \oplus a') - (a + a') : a, a' \in A\}.$$

□

This ring of virtual bundles is our object of study.

Definition 2. *The ring of isomophism classes of complex virtual bundles over X is the* ***K-theory*** *of X, denoted by $K(X)$. The association $X \mapsto K(X)$ is a contravariant functor from topological spaces to rings. [There is also the real K-theory of X, denoted by $KO(X)$.]*

K-theory is functorial in X as follows. Given $X \xrightarrow{f} Y$ and a vector bundle $E \xrightarrow{p} Y$, we have the pullback $f^*E \xrightarrow{q} X$,

$$f^*E = X \times_Y E = \{(x,e) \in X \times E : p(e) = f(x)\},\ q(x,e) = x,$$

a vector bundle over X with fiber $q^{-1}(x) = p^{-1}(f(x))$. That the conditions for a vector bundle are satisfied can be checked by pulling back a local trivialization of $E \xrightarrow{p} Y$. It is straightforward to verify that this gives a ring homomorphism $K(Y) \xrightarrow{f^*} K(X)$.

In fact, K factors through the homotopy category.

Proposition 2 ([VBKT] 1.6, [AGP] 8.4.4.)**.** *If $f_0, f_1 : A \to B$ are homotopic through $H : A \times I \to B$ and $p : E \to B$ is a vector bundle, then $f_0^*(E) \cong f_1^*(E)$ (assuming A is compact Hausdorff).*

Proof. Considering the pullback $q : H^*(E) \to A \times I$, we want to show that

$$f_0^*(E) = q^{-1}(A \times \{0\}) \cong q^{-1}(A \times \{1\}) = f_1^*(E).$$

- First we note that any bundle $E \xrightarrow{p} X \times I$ is trivial if the restrictions over $X \times [0,1/2]$, $X \times [1/2,1]$ are trivial. If these restrictions are E_0, E_1 with local trivializations

  $$E_0 \xrightarrow{h_0} X \times [0,1/2] \times \mathbb{C}^n,\ E_1 \xrightarrow{h_1} X \times [1/2,1] \times \mathbb{C}^n$$

  then we have the isomorphism $h_1^{-1} \circ h_0 : X \times \{1/2\} \times \mathbb{C}^n \to X \times \{1/2\} \times \mathbb{C}^n$ given by $(x,1/2,v) \mapsto (x,1/2,g(v))$ for some linear g, and the map

  $$X \times I \times \mathbb{C}^n \to B,\ (x,t,v) \mapsto \begin{cases} h_0(x,t,v) & 0 \le t \le 1/2 \\ h_1(x,t,g(v)) & 1/2 \le t \le 1 \end{cases}$$

  is a trivialization of E.

- Secondly, we note that any vector bundle $E \xrightarrow{p} B \times I$ has a local trivialization of the form $U_i \times I$ for an open cover U_i of B. Given $x \in X$, there is a collection $\{U_{x,i}\}_{i=1}^n$ of neighborhoods of x and a partition $0 = t_0 < \cdots < t_n = 1$ such that the restriction of E over each of $U_{x,i} \times [t_{i-1}, t_i]$ is trivial (using compactness of I). By the previous point, the restriction of E over $U_x \times I$ is trivial where $U_x = \cap_{i=1}^n U_{x,i}$.

Now take any vector bundle $E \xrightarrow{p} X \times I$, and let $X = \cup_{i=1}^n U_i$ be such that E is trivial over U_i with a partition of unity ϕ_i subordinate to the U_i. Let $\psi_i = \sum_{j=1}^i \phi_j$, $X_i \subseteq X \times I$ the graph of ψ_i (homeomorphic to X), and $E_i \xrightarrow{p_i} X_i$ the restriction of E over X_i. Because E is trivial over $U_i \times I$, the homeomorphisms $h_i : X_i \to X_{i-1}$ lift to homeomorphisms $\tilde{h}_i : E_i \to E_{i-1}$ that are the identity outside of $p^{-1}(U_i \times I)$

$$\tilde{h}_i(x, \psi_i(x), v) = (x, \psi_{i-1}(x), v),\ (x, \psi_i(x), v) \in U_i \times I \times \mathbb{C}^n \cong p^{-1}(U_i \times I).$$

The composition $\tilde{h}_1 \circ \cdots \circ \tilde{h}_n$ is a homeomorphism $E|_{X \times \{0\}} \cong E|_{X \times \{1\}}$. ∎

Some examples:

- $K(pt.) = \mathbb{Z}$ (a vector bundle over a point is a vector space, determined up to isomorphism by its dimension).

- We'll see that $K(S^2) = \mathbb{Z}[H]/(H-1)^2$ generated by the canonical bundle $H = \{(L, x) : L \in P^1(\mathbb{C}), x \in L\}$, viewing S^2 as the collection of complex lines L through the origin in $\mathbb{C}^2$. We can at least see that the relation $H \otimes H \oplus 1 = H \oplus H$ holds using the clutching functions
$$f(z) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix},\ g(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$
on the intersection of the affine open cover. The canonical bundle H is trivial over the two copies of the plane $[1 : w]$ and $[z : 1]$ $(w = 1/z)$. Over their intersection, a point $(\lambda, \lambda w) \in [1 : w]$ is mapped to the point $(\lambda z, \lambda) \in [z : 1]$ via mulitiplication by z. Hence the cluthing for $H \otimes H \oplus 1$ and $H \oplus H$ are as stated above. The map $\alpha_t : [0, 1] \to GL_2(\mathbb{C})$ given by
$$\alpha_t = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}$$
is a path in $GL_2(\mathbb{C})$ between
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
so that we have a homotopy $f \simeq g$ between the clutching functions for the two bundles $H \otimes H \oplus 1$, $H \oplus H$ defined by
$$\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \alpha_t \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \alpha_t.$$
Hence the two bundles are isomorphic and we have a homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$.

We also have a **reduced K-theory**. Over pointed spaces, the inclusion $\{pt.\} \to X$ induces a map $K(X) \to \mathbb{Z}$ sending a bundle to the dimension of its fiber over the connected component of the basepoint. The reduced K-theory of X, $\widetilde{K}(X)$, is the kernel of this ring map. The inclusion of a basepoint and the constant map to the base point induce a splitting $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$.

2 Classification of vector bundles

Definition 3. *The* **Grassmann manifold** $G_n(\mathbb{C}^N)$ *for* $n \leq N$ *is the collection of* n*-dimensional complex vector subspaces of* $\mathbb{C}^N$*, topologized as* $U_N/(U_n \times U_{N-n})$ *(after choosing an orthonormal basis for* $\mathbb{C}^N$*, the unitary group acts transitively on* n*-planes, with stabilizer those matrices fixing the* n*-plane and its orthogoal complement). Taking the direct limit under the inclusions* $G_n(\mathbb{C}^N) \to G_n(\mathbb{C}^{N+1})$*, we obtain* $G_n(\mathbb{C}^\infty)$*. These Grassmann manifolds come with* **canonical bundles**
$$E_n(\mathbb{C}^N) \to G_n(\mathbb{C}^N),\ E_n(\mathbb{C}^N) = \{(P, v) : P \in G_n(\mathbb{C}^N), v \in p\},$$
$$E_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty),\ E_n(\mathbb{C}^\infty) = \{(P, v) : P \in G_n(\mathbb{C}^\infty), v \in p\}.$$

The $G_n(\mathbb{C}^N)$ are closed complex manifolds of complex dimension $n(N-n)$. One useful representation is as $n \times (N-n)$ matrices (the rows spanning the n-plane) modulo the (left) GL_n action. Each n-plane then has a unique representative in reduced row echelon form, and grouping these representatives by their pivot positions gives a cell structure on $G_n(\mathbb{C}^N)$. [For a discussion of the characteristic maps, see [VBKT] proposition 1.17.] By introducing a column of zeros, we have a subcomplex $G_n(\mathbb{C}^N) \subseteq G_n(\mathbb{C}^{N+1})$. For example, a 4-cell in $G_3(\mathbb{C}^5)$ included into $G_3(\mathbb{C}^6)$ could look like

$$\begin{pmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & * & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}.$$

The elements $G_n(\mathbb{C}^\infty)$ can be represented by $n \times \infty$ matrices in reduced row echelon form with only finitely many non-zero entries. By introducing a row of all zeros and single 1, we have a subcomplex $G_n(\mathbb{C}^\infty) \subseteq G_{n+1}(\mathbb{C}^\infty)$. For instance, the 4-cell above becomes (after inclusion into $G_3(\mathbb{C}^\infty)$ then inclusion into $G_4(\mathbb{C}^\infty)$)

$$\begin{pmatrix} \dots & 0 & 1 & * & 0 & 0 & * & 0 \\ \dots & 0 & 0 & 0 & 1 & 0 & * & 0 \\ \dots & 0 & 0 & 0 & 0 & 1 & * & 0 \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 4. *The **classifying spaces** of the finite and infinite unitary groups are*

$$BU_n = \cup_N G_n(\mathbb{C}^N) = G_n(\mathbb{C}^\infty), \ BU = \cup_n BU_n$$

with respect to the maps coming from the previous paragraph.

We can begin to relate K-theory to homotopy and generalized cohomology with the following

Proposition 3 ([VBKT] 1.16, [AGP] 8.5.13)**.** *There is a bijection between homotopy classes of maps from X to the Grassmannian of n-planes in $\mathbb{C}^\infty$ and ismophism classes of rank n vector bundles over X (assuming X is compact Hausdorff):*

$$[X, BU_n] \to \mathrm{Vect}^n_{\mathbb{C}}(X), \ [f] \mapsto f^*(E_n(\mathbb{C}^\infty)).$$

Proof. Suppose $E \xrightarrow{p} X$ is an n-dimensional vector bundle. First, we note that an isomorphism $E \cong f^*(E_n(\mathbb{C}^\infty))$ is equivalent to a map $g : E \to \mathbb{C}^\infty$ that is a linear injection on each fiber (such a g is what [AGP] calls a *Gauss map*). If we have such an isomorphism, consider the following diagram

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[dr, "p"'] & f^*E_n(\mathbb{C}^\infty) \arrow[r, "\tilde{f}"] \arrow[d] & E_n(\mathbb{C}^\infty) \arrow[r, "\pi"] \arrow[d] & \mathbb{C}^\infty \\ & X \arrow[r, "f"] & BU_n & \end{tikzcd}$$

(where $\pi(P, v) = v$). The composition $g = \tilde{f} \circ \pi$ is a linear injection on each fiber since both $\tilde{f}$ and π are. Conversely, given such a g, define $f(x) = g(p^{-1}(x)) \in BU_n$ to get an isomorphism $E \cong f^*E_n(\mathbb{C}^\infty)$

$$v \in p^{-1}(x) \mapsto ((f(x), g(v)), x) \in f^*E_n(\mathbb{C}^\infty).$$

- [Surjectivity.] Let $E \xrightarrow{p} X$ be an n-dimensional vector bundle, $X = \cup_{i=1}^m U_i$ a finite open cover over which E is trivial, and $\{\phi_i\}_i$ a partition of unity subordinate to the cover. For $(x, v) \in p^{-1}(U_i)$, the map $(x, v) \mapsto \phi_i(x)v \in \mathbb{C}^n$ extends to a map $g_i : E \to \mathbb{C}^n$ that is zero outside of $p^{-1}(U_i)$. Putting these together gives a map $g : E \to (\mathbb{C}^n)^m \subseteq \mathbb{C}^\infty$ that is a linear injection on each fiber.

- [Injectivity.] If $E \cong f_0^* E_n(\mathbb{C}^\infty), f_1^* E_n(\mathbb{C}^\infty)$, let $g_0, g_1 : E \to \mathbb{C}^\infty$ be maps which are linear injections on each fiber $p^{-1}(x)$ as discussed at the beginning of the proof. We want to show that $g_0 \simeq g_1$ via g_t that are linear injections on each fiber, so that $f_0 \simeq f_1$ via $f_t(x) = g_t(p^{-1}(x))$. Let $A_t, B_t : \mathbb{C}^\infty \to \mathbb{C}^\infty$ be the homotopies

$$A_t(z_i) = (1-t)(z_1, z_2, z_3, \dots) + t(z_1, 0, z_2, 0, z_3, 0, \dots),$$
$$B_t(z_i) = (1-t)(z_1, z_2, z_3, \dots) + t(0, z_1, 0, z_2, 0, z_3, 0, \dots).$$

  We have $g_0 \simeq A_1 \circ g_0 =: G_0$ (putting g_0 into the odd coordinates), $g_1 \simeq B_1 \circ g_1 =: G_1$ (putting g_1 into the even coordinates), and $G_0 \simeq G_1$ via $(1-t)G_0 + tG_1$ (all through maps that are linear injections on each fiber).

□

Here is another splitting $K(X)$ (X compact) that agrees with $K(X) = \widetilde{K}(X) \oplus \mathbb{Z}$ when X is connected. The function $d : \mathrm{Vect}_{\mathbb{C}}(X) \to [X, \mathbb{N}]$, $d_E(x) = \dim p^{-1}(x)$, extends to $\bar{d} : K(X) \to [X, \mathbb{Z}]$. Denote the kernel of $\bar{d}$ by $\widehat{K}(X)$. We have $K(X) = \widehat{K}(X) \oplus [X, \mathbb{Z}]$ (given $f : X \to \mathbb{Z}$, X decomposes as a finite union $X = \cup_n f^{-1}(n)$ and we take the trivial virtual bundles of dimension n over $f^{-1}(n)$ to obtain a splitting).

Denote by $\mathrm{Vect}^s_{\mathbb{C}}(X)$ the colimit of the $\mathrm{Vect}^k_{\mathbb{C}}(X)$ under the inclusions $[E] \mapsto [E \oplus \epsilon]$ where ϵ is the trivial one-dimensional bundle over X.

Lemma 1. *If X is compact Hausdorff and $E \xrightarrow{p} X$ is a complex vector bundle, then there is another bundle $E' \xrightarrow{p'} X$ such that $E \oplus E'$ is trivial.*

Proof. Let $X = \cup U_i$ be a finite cover such that $p^{-1}(U_i) \cong U_i \times \mathbb{C}^{n_i}$ and let $\{\phi_i\}_i$ be a partition of unity subordinate to $\{U_i\}_i$. Define $g_i : E \to \mathbb{C}^{n_i}$ by $g(x, v) = \phi_i(x)v$ for $(x, v) \in U_i \times \mathbb{C}^{n_i}$ (so g_i is zero outside $p^{-1}(U_i)$) and use these as coordinates for a function $g : E \to \bigoplus_i \mathbb{C}^{n_i}$. This g is a linear injection on each fiber and the map $G : E \to X \times \bigoplus_i \mathbb{C}^{n_i}$, $G(e) = (p(e), g(e))$ gives E as a sub-bundle of a trivial bundle. Taking the orthogonal complement fiberwise gives the desired E'. □

Finally, we have the following

Theorem 1 ([AGP] 9.4.6-9.4.9)**.** *If X is compact, then*

$$\widehat{K}(X) \cong \mathrm{Vect}^s_{\mathbb{C}}(X) \cong [X, BU],$$

from which it follows that

$$K(X) \cong [X, BU \times \mathbb{Z}].$$

Proof. For each $k \geq 0$ we have a map

$$\phi_k : \mathrm{Vect}^k_{\mathbb{C}}(X) \to \widehat{K}(X),\ \phi_k([E]) = [E] - [\epsilon^k],$$

commuting with the maps defining the colimit $\mathrm{Vect}^s_{\mathbb{C}}(X)$, hence a map $\phi : \mathrm{Vect}^s_{\mathbb{C}}(X) \to \widehat{K}(X)$. We want to show that ϕ is a semigroup isomorphism (hence an isomorphism of abelian groups). For $[E_1] - [E_2] \in \widehat{K}(X)$, there are E_i' such that $E_i \oplus E_i' \cong \epsilon^n$ is trivial by the previous lemma. We have

$$[E_1 \oplus E_2'] - [\epsilon^n] = [E_1] - [E_2] \in \widehat{K}(X),$$

so that $d([E_1 \oplus E_2']) = d([\epsilon^n]) = n$ is constant and $\phi_n([E_1 \oplus E_2']) = [E_1] - [E_2]$. Hence ϕ is surjective.

For injectivity of ϕ, if $[E] - [\epsilon^k] = [E'] - [\epsilon^l]$ in $\widehat{K}(X)$ (note that every element of the direct limit has a representative in one of the $\mathrm{Vect}^k_{\mathbb{C}}(X)$), then there is an n such that $E \oplus \epsilon^{l+n} \cong E' \oplus \epsilon^{k+n}$. [We know that there is some bundle F such that $E \oplus \epsilon^l \oplus F \cong E' \oplus \epsilon^k \oplus F$, then adding some F' such that $F' \oplus F \cong \epsilon^n$ gives what we want.] Hence E and E' represent the same element in $\mathrm{Vect}^s_{\mathbb{C}}(X)$.

Finally, we have

$$\widehat{K}(X) \cong \mathrm{Vect}^s_{\mathbb{C}}(X) = \mathrm{colim}_k \, \mathrm{Vect}^k_{\mathbb{C}}(X) \cong \mathrm{colim}_k [X, BU_k] \cong [X, BU],$$

the last isomorphism since X is compact, and the isomorphism between the colimits commuting with pullback (the inclusion $i_k : BU_k \to BU_{k+1}$ satisfies $i_k^*(E_k(\mathbb{C}^\infty)) = E_k(\mathbb{C}^\infty) \oplus \epsilon$ and the inclusion $\mathrm{Vect}^k_{\mathbb{C}}(X) \to \mathrm{Vect}^{k+1}_{\mathbb{C}}(X)$ takes $[E]$ to $[E \oplus \epsilon]$). □

3 Bott Periodicity and K-theory as a generalized cohomology theory

References for the following section are [AGP] chapters 9 and 12.

Definition 5. *[AGP] 12.1.4 An additive* ***reduced cohomology theory*** $\widetilde{E}^*$ *is a sequence of contravariant functors* $\{\widetilde{E}^n\}_{n\in\mathbb{Z}}$ *from the category of pointed topological spaces to abelian groups with the following properties:*

- *(suspension isomorphisms)* $\widetilde{E}^n(\Sigma X) \cong \widetilde{E}^{n-1}(X)$*, natural in* X*,*
- *(homotopy invariance) if* $f \simeq g : (X, x_0) \to (Y, y_0)$ *then* $f^* = g^* : \widetilde{E}^*(X) \to \widetilde{E}^*(Y)$*,*
- *(exactness) for each pair* (X, A) *there is an exact sequence*

$$\widetilde{E}^n(X \cup CA) \to \widetilde{E}^n(X) \to \widetilde{E}^n(A)$$

  (maps induced by inclusions, CA *the reduced cone on* A*),*
- *(additivity)* $\widetilde{E}^n(\vee_\alpha X_\alpha) \cong \prod_\alpha \widetilde{E}^n(X_\alpha)$*, induced by the inclusions.*

Representability of cohomology theories are mediated by the following objects (proofs definitely omitted).

Definition 6. *An* Ω***-presepectrum*** *is a sequence of pointed spaces* $\{E_n\}_{n\in\mathbb{Z}}$ *and weak homotopy equivalences* $\epsilon_n : E_n \to \Omega E_{n+1}$*.*

The hard direction of the following theorem follows from Brown's representability theorem, [AGP] theorem 12.2.22.

Theorem 2 ([AGP] 12.3.2, 12.3.3)**.** *An* Ω*-prespectrum determines an additive reduced cohmology theory* $\widetilde{E}^*$ *on the category of pointed topological spaces via*

$$\widetilde{E}^n = [X, E_n]_*.$$

Conversely, an additve reduced cohomology theory $\widetilde{E}^*$ *on the category of pointed CW-complexes determines an* Ω*-prespectrum* $\{E_n\}_{n\in\mathbb{Z}}$ *such that*

$$\widetilde{E}^n = [X, E_n]_*.$$

Theorem 3 (Bott periodicity, [AGP] 9.5.1 and appendix B). *There is a homotopy equivalence*

$$\Omega^2 BU \simeq BU \times \mathbb{Z}.$$

[The homotopy groups of the classifying space BU are periodic with period two

$$\pi_{i+2}(BU) = \pi_i(\Omega^2 BU) = \pi_i(BU \times \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \pi_i(BU) = 0 & i \geq 1, \end{cases}$$

hence "periodicity."]

Bott periodicity shows that we have an Ω-prespectrum consisting of

$$E_{2n} = BU \times \mathbb{Z},\ E_{2n+1} = \Omega BU,$$

hence a reduced cohomology theory

$$\widetilde{E}^n(X) = [X, E_n]_* =: \widetilde{K}^n(X)$$

which agrees with our previous definition on compact Hausdorff X.

3.1 Another version of Bott periodicity

Given spaces X and Y, the projections $p_1, p_2 : X \times Y \to X, Y$ give a multiplication

$$K(X) \otimes K(Y) \xrightarrow{\mu} K(X \times Y),\ \mu(a \otimes b) = p_1^*(a) p_2^*(b)$$

or in the reduced case

$$\widetilde{K}(X) \otimes \widetilde{K}(Y) \xrightarrow{\tilde{\mu}} \widetilde{K}(X \wedge Y).$$

Taking $Y = S^2$ we get isomorphisms.

Theorem 4 ([VBKT] 2.2, [Bott] appendix I). *We have isomorphisms via the external products μ, $\tilde{\mu}$ defined above:*

$$K(X) \otimes K(S^2) \xrightarrow{\cong} K(X \times S^2),\ \widetilde{K}(X) \otimes \widetilde{K}(S^2) \xrightarrow{\cong} \widetilde{K}(X \wedge S^2) \cong \widetilde{K}(\Sigma^2 X).$$

Knowledge that $\widetilde{K}(S^2) \cong \mathbb{Z}$ (generated by $H - 1$) gives another version of Bott periodicity.

Theorem 5 (Bott periodicity, [VBKT] 2.11). *There is an isomorphism*

$$\widetilde{K}(X) \xrightarrow{\otimes(H-1)} \widetilde{K}(X) \otimes \widetilde{K}(S^2) \xrightarrow{\tilde{\mu}} \widetilde{K}(X \wedge S^2) \cong \widetilde{K}(\Sigma^2 X),$$

so that the reduced K-theory of X is 2-periodic under suspension.

4 An application of K-theory

Although proved earlier using other techniques, there is a (relatively) simple proof of the following theorem using K-theory, cf. [VBKT] section 2.3 or [AGP] chapter 10.

Theorem 6. *The only spheres that are H-spaces are*

$$S^0,\ S^1,\ S^3,\ S^7,$$

coming from the algebra structure on the norm one elements of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$.

The proof uses the existence of the **Adams operations** in K-theory and the **splitting principle**.

Definition 7 (Adams operations)**.** *For a compact space X and $k \geq 0$ there are $\psi^k : K(X) \to K(X)$ satisfying:*

- $f^*\psi^k = \psi^k f^*$ *for* $f : X \to Y$*,*
- $\psi^k(L) = L^k$ *for line bundles* L*,*
- $\psi^k\psi^l = \psi^{kl}$*,*
- $\psi^p(x) \equiv x^p (\mathrm{mod} p)$*.*

Theorem 7 (splitting principle)**.** *Given a vector bundle $E \xrightarrow{p} X$ over a compact space, there is a compact space $P(E)$ and a map $P(E) \xrightarrow{f} X$ such that $f^*P(E)$ splits as a sum of line bundles and $f^* : K(X) \to K(F(E))$ is injective. [Here $P(E)$ is a projective bundle over X - remove the zero section from $E \xrightarrow{p} X$ and quotient by the $\mathbb{C}^\times$ action or take the space of lines in each fiber.]*

5 Other flavors of K-theory

We briefly mention aspects of other incarnations of K-theory.

5.1 Real K-theory

Here is the version of Bott periodicity for real vector bundles.

Theorem 8. *There is a homotopy equivalence*

$$\Omega^8 BO \simeq BO \times \mathbb{Z}.$$

Hence the cohomology theory associated to KO has period eight instead of period two.

5.2 Algebraic K-theory

The following theorem of Swan shows links to other areas of mathematics.

Theorem 9. *Suppose X is compact Hausdorff and let $R = C(X; \mathbb{R})$ be the ring of real valued continuous functions on X. If $E \xrightarrow{p} X$ is an $\mathbb{R}$-vector bundle, then the ring of global sections $\Gamma(X) = \{s : X \to E : p \circ s = 1_X\}$ is a finitely generated projective R-module and every finitely generated projective R-module arises in this fashion.*

Focusing on finitely generated projective modules over a given ring leads to algebraic K-theory.

Definition 8. *Let R be a ring and define $K_0(R)$ to be the Grothendieck group of the monoid of finitely generated projective R-modules under direct sum. $K_0(R)$ becomes a ring with product extending the tensor product of modules. K_0 is a covariant functor from Rings to Rings. Every ring has a map $\mathbb{Z} \to R$ and the cokernel of the induced map gives the reduced group, $\widetilde{K}_0(R)$.*

As an example, if R is a number ring, $\widetilde{K}_0(R)$ is the class group of R. There are definitions for higher K-groups, but we will not discuss them here.

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