EG and BG

Leo Herr
CU Boulder PhD Program

Simplicial sets are supposedly useful, and we will now see an application. Let π be an abelian group. View it as a discrete topological group whenever necessary by considering the underlying set as a discrete topological space. We want $K(\pi, n)$:

$$\pi_q K(\pi, n) = \begin{cases} \pi & q = n \\ 0 & q \neq n \end{cases}$$

If G is a topological group, we describe a functor $B : \text{Topgrp} \to \text{Topgrp}$, to be thought of as Ω^{-1}, so that

$$K(\pi, n) = B^n \pi$$

($B^n\pi$ means $B \circ B \circ \cdots \circ B\pi$, and we consider π as a discrete topological group in order to apply B for the first time.)
Recall Simplicial Sets from Paul's talk (and corresponding notes online). We work with an equivalent definition, which stems from applying exercise 1.6 in his notes to digest the notion of a presheaf on Δ to the point we can work with it. To the notion of a presheaf $X : \Delta^{op} \to (Set)$, we keep track of the sets $X_n := \text{Hom}(\Delta^n, X)$ and the identities they must follow.
A simplicial set $\{X_n\}_{n\in\mathbb{N}}$ is:

- For each $n \in \mathbb{N}$, a set X_n. (Including $0 \in \mathbb{N}$.)
- For each $n \geq 0$, $0 \leq i \leq n$, functions called "ith face maps" : $d_i : X_n \to X_{n-1}$.
- For each $n \geq 0$, $0 \leq i \leq n$, functions called "ith degeneracy maps": $s_i : X_n \to X_{n+1}$

satisfying the simplicial identities, which regulate how face and degeneracy maps compose:

- $d_i \circ d_j = d_{j-1} \circ d_i$ if $i < j$

- $d_i \circ s_j = s_{j-1} \circ d_i$ if $i < j$
- $d_i \circ s_j = id$ if $i = j$ or $i = j + 1$
- $d_i \circ s_j = s_j d_{i-1}$ if $i > j + 1$
- $s_i \circ s_j = s_{j+1} \circ s_i$ if $i \leq j$.

A simplicial set may be thought of as the singular simplicial set corresponding to a space with X_0-many points, X_1-many edges, X_2-many faces, and so on. This is for the sake of intuition only, and we may have simplicial objects in whatever category. For example, a simplicial ***space*** $\{X_n\}_{n \in \mathbb{N}}$ is:

- For each $n \in \mathbb{N}$, a ***space*** X_n.
- For each $n \geq 0$, $0 \leq i \leq n$, ***continuous maps*** called "ith face maps" : $d_i : X_n \to X_{n-1}$.
- For each $n \geq 0$, $0 \leq i \leq n$, ***continuous maps*** called "ith degeneracy maps": $s_i : X_n \to X_{n+1}$

satisfying the simplicial identities as above.
Simplicial objects are like chain complexes, but with more arrows in between the objects. Indeed, there is an explicit and clever dictionary between non-negatively graded chain complexes and simplicial abelian groups called the "Dold-Kan Correspondence."

Exercise 1

Define simplicial ***thinghood***.

Geometric realization builds a bona fide topological space out of a simplicial set $\{X_n\}_{n \in \mathbb{N}}$:

- The ingredients are X_n-many n-simplices
- They are glued together according to the face and degeneracy maps.

The details of geometric realization are left to the Appendix. This is not because they're unimportant, but because they're difficult and notationally dense.

The simplicial *spaces* $\{B_nG\}$ and $\{E_nG\}$ are defined as follows:

$$B_nG := G^{\times n}$$
$$E_nG := B_nG \times G = G^{\times n+1}$$

There is a projection map $p_n : E_nG \to B_nG$ onto the first n coordinates.
Visually (drawing only the face maps):

$$B_*G : \cdots G \times G \times G \overset{\rightrightarrows}{\rightrightarrows} G \times G \overset{\rightarrow}{\rightrightarrows} G \rightrightarrows pt$$

and

$$E_*G : \cdots G \times G \times G \times G \overset{\rightrightarrows}{\rightrightarrows} G \times G \times G \overset{\rightarrow}{\rightrightarrows} G \times G \rightrightarrows G$$

For E_*G, the face maps:

$$d_i(g_1, \ldots, g_{n+1}) := (g_1, \ldots, g_i \cdot g_{i+1}, \ldots, g_{n+1})$$

except if $i = 0$ – then,

$$d_0(g_1, \ldots, g_{n+1}) := (g_2, \ldots, g_{n+1})$$

the degeneracy maps:

$$s_i(g_1, \ldots, g_{n+1}) := (g_1, \ldots, g_{i-1}, e, g_i, \ldots g_{n+1})$$

where $e \in G$ is the identity.
For B_*G, the maps are the same except without g_{n+1}, and

$$d_n(g_1, \ldots, g_n) := (g_1, \ldots, g_{n-1})$$

Exercise B

Check simplicial identities for E_*G, B_*G. Check that p_n commutes with degeneracy and face maps to give a map of simplicial spaces: $p_* : E_*G \to B_*G$

Let G act on E_*G on the right:

$$(g_1, \ldots, g_{n+1}).g := (g_1, \ldots, g_{n+1} \cdot g)$$

Then check that the d_i's for E_*G are G-equivariant (Exercise B'). E_*G is thus a "simplicial G-space" (see Exercise 1).

The G-action just changes the last coordinate arbitrarily, so quotienting out by the G-action just forgets the last coordinate.

$$E_nG/G := B_nG \times G/G = B_nG$$

These simplicial sets have the properties of $K(\pi, n)$'s already, and we only make topological spaces out of them for familiarity.
Geometrically Realize each of the simplicial sets we've constructed to get spaces:

$$EG := |E_*G|$$

$$BG := |B_*G|$$

$$p := |p_*| : EG \to BG$$

As a corollary of 2.17 in Paul's notes, $|\cdot|$ commutes with colimits – in particular, it should commute with taking quotients, suitably interpreted. Then $EG/G = |E_*G/G| = |B_*G| = BG$.
In fact, if G is nondegenerately based,

$$\begin{tikzcd} G \arrow[r, hook, "\subseteq"] & EG \arrow[d] \\ & BG \end{tikzcd}$$

is a fiber bundle. Even better, $EG \simeq pt$ is contractible.

Remark 0.1. Sketch of a justification: According to Paul's reinterpretation of a homotopy class in terms of simplicial sets (so-called "simplicial homotopy groups"), we need only check that there exists a lift in the following diagram, for any upper horizontal map representing a homotopy class.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & E_*G \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & pt \end{tikzcd}$$

Lifting this diagram is left as an exercise – indeed, this entire remark helps to justify J. P. May's statement that EG is contractible.
Observe the lower right corner is vacuous. The lifting of the diagram simply says that $\pi_n EG \to \pi_n pt$ is an injection, and all nonzero homotopy groups of pt are zero. Moreover, EG is connected, so EG is weakly equivalent to a point. Furthermore, it is homotopy equivalent to a point, since it's a CW complex.

So the Long Exact Sequence of homotopy says:

$$\cdots \pi_{q+1}EG \to \pi_{q+1}BG \to \pi_q G \to \pi_q EG \to \cdots$$

and the outer two groups are zero by contractibility. Therefore:

$$\pi_q B^n \pi = \pi_{q-1} B^{n-1} \pi = \cdots = \pi_{q-n} \pi = \begin{cases} \pi & q = n \\ 0 & q \neq n \end{cases}$$

Appendix : The Details of Geometric Realization for Simplicial Spaces

What does $|E_* G|$ mean for simplicial spaces? The motto to keep in mind is the following:
"Add n-simplices labelled by $E_n G$; glue via face and boundary maps."
Glue along the following:

$$\begin{tikzcd} & S_n X \times \Delta^n \\ S_n X \times \Delta^{n+1} \arrow[ur, "id \times \sigma_i"] \arrow[dr, "s_i \times id"'] & \\ & S_{n+1} X \times \Delta^{n+1} \end{tikzcd} \qquad \begin{tikzcd} & S_n X \times \Delta^n \\ S_n X \times \Delta^{n-1} \arrow[ur, "id \times \delta_i"] \arrow[dr, "d_i \times id"'] & \\ & S_{n-1} X \times \Delta^{n-1} \end{tikzcd}$$

where σ_i, δ_i correspond to the coface, codegeneracy maps on Δ^n's Paul mentioned. By "glue along," we mean to identify the two images under the two maps of each point in the space on the left of each diagram.
More explicitly,

$$E_* G := \amalg S_n X \times \Delta^n / \sim$$

where $\sim$ equates:

$$(f, \delta_i u) \sim (d_i f, u)$$

and

$$(f, \sigma_i u) \sim (s_i f, u)$$

for $f \in S_n X$, $u \in \Delta^n$.

Bibliography

Paul's Course Notes, classes 30 and 31
J. P. May's *Concise Algebraic Topology*