RATIONAL HOMOTOPY THEORY

Contents of these notes are based on:

- *Rational Homotopy Theory*, Y. Felix, S. Halperin, J.C. Thomas
- *Rational Homotopy Theory: A Brief Introduction*, K. Hess
- *Differential Forms in Algebraic Topology*, R. Bott, L.W. Tu

The Big Picture

It is well-known that the homotopy groups of spaces with simple CW-decompositions are difficult to compute. Since it is known that $\pi_n(X)$ is a finitely generated abelian group for $n \geq 2$, the computation of $\pi_n(X)$ can be broken into two parts: computing the rank of $\pi_n(X)$ and computing the torsion of $\pi_n(X)$. The second step is in general quite difficult. However, there is an elegant method for computing the first part for a wide class of spaces. The idea is to study $\pi_n^{\mathbb{Q}}(X) := \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is a torsion-free abelian group with the same rank as $\pi_n(X)$. Of course, this definition is not useful as a computational tool; if we knew $\pi_n(X)$, we wouldn't be interested in $\pi_n^{\mathbb{Q}}(X)$. However, using techniques pioneered by D. Sullivan and D. Quillen, $\pi_n^{\mathbb{Q}}(X)$ can often be computed from the cohomology of X. We outline the basic concepts of Sullivan's approach to rational homotopy theory and apply it to the computation of the rational homotopy groups of spheres.

The Theory

We assume that all spaces are simply connected and compactly generated.

Def: A space X is called *rational* if each of its homotopy groups is a $\mathbb{Q}$-vector space. A continuous map $f : X \to X_0$ is called a *rationalization* of X if $\pi_* f \otimes \mathbb{Q}$ is an isomorphism.

Theorem:(Sullivan) For every space X there exists a relative CW-complex (X_0, X) with no zero cells or one cells such that $X \to X_0$ is a rationalization of X. Furthermore, if Y is a rational space, then any map $X \to Y$ can be extended uniquely up to homotopy to a map $X_0 \to Y$.

Def: The *rational homotopy type* of a space X is the weak homotopy type of its rationalization X_0. A map $f : X \to Y$ is a *rational homotopy equivalence* if $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism.

Def: A *commutative differential graded algebra* (CDGA) over $\mathbb{Q}$ is a cochain complex (A^*, d) over $\mathbb{Q}$ with cochain maps $\iota : \mathbb{Q} \to (A^*, d)$ (the unit map) $\mu : (A^*, d) \otimes_{\mathbb{Q}} (A^*, d) \to (A^*, d)$ (the multiplication map) such that μ is graded commutative and associative and ι is a left and right identity for μ. Then μ and d obey the graded Leibniz rule. A CDGA A is *simply connected* if $A^0 = \mathbb{Q}$ and $A^1 = 0$. Given a non-negatively graded $\mathbb{Q}$-vector space $V = \oplus_i V^i$, the free CDGA generated by V is $\Lambda V = S[\oplus_{ieven} V^i] \otimes_{\mathbb{Q}} E[\oplus_{iodd} V^i]$, the tensor product of the symmetric algebra on the even generators and the exterior algebra on the odd generators. A *minimal Sullivan algebra* M for a CDGA A is an inclusion of CDGAs $(A, d) \to (A \otimes_{\mathbb{Q}} \Lambda V, d)$ where V is a graded vector space with basis $\{v_i\}_{i \in J}$ with J a well ordered set such that $dv_i \in A \otimes \Lambda V_{<i}$ for all $i \in J$ and such that $i < j$ implies $\deg v_i < \deg v_j$.

Theorem: There is an equivalence of categories between the category of rational homotopy types of simply connected spaces of finite rational type and isomorphism classes of minimal Sullivan algebras.

Theorem: The number of generators of degree n in the minimal Sullivan algebra for X is the rank of $\pi_n^{\mathbb{Q}}(X)$.

Def: A *minimal model* for a CDGA A is a free CDGA M such that there is a quasi-isomorphism $f : M \to A$ and such that for each generator x of M, either $dx = 0$ or $dx \in M^{\geq 1} \cdot M^{\geq 1}$.

Theorem: In the case that X is a smooth manifold, there is a quasi-isomorphism of CDGAs over $\mathbb{R}$ between the minimal Sullivan algebra of X tensored with $\mathbb{R}$ and the minimal model for the algebra of de Rham forms on X. Therefore the number of generators for the minimal model of the de Rham algebra is the same as the number of generators of the minimal algebra of X.

Example

We compute the rational homotopy groups of spheres with dimension at least 2. In the case of a sphere of odd dimension $2n-1$, the de Rham cohomology of S^{2n-1} is an exterior algebra with a single generator in degree $2n-1$, which is a minimal model for itself. Therefore $\pi_k^{\mathbb{Q}}(S^{2n-1})$ is zero except when $k = 2n-1$, in which case $\pi_{2n-1}^{\mathbb{Q}}(S^{2n-1}) \cong \mathbb{Q}$.

In the case of a sphere of even dimension $2n$, the de Rham cohomology ring is $\mathbb{R}[\omega]/(\omega^2)$, where ω has degree $2n$. To construct a minimal model, we need a generator x of degree $2n$ to map to ω with $dx = 0$. Since x^2 need not be zero (x has even degree), we also need a generator y of degree $4n-1$ so that $dy = x^2$. Since y has odd degree, $y^2 = 0$. Using the Leibniz rule, we see that $x^n = d(yx^{n-2})$ and $d(x^n) = 0$, so that no other products of generators contribute nontrivially to cohomology. It follows that a minimal model for the de Rham algebra of S^{2n} has two generators, one in degree $2n$ and one in degree $4n-1$, so that the rational homotopy groups $\pi_k^{\mathbb{Q}}(S^{2n})$ are 0 except when $k = 2n$ or $4n-1$, in which case $\pi_k^{\mathbb{Q}}(S^{2n}) \cong \mathbb{Q}$.