MATH 6280 - CLASS 9

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Cofibrations and the HEP - Continued

Proposition 1.1. *The pushout of a cofibration is cofibration. That is, for $A \xrightarrow{i} X$ a cofibration and $f : A \to Y$ be a map, then $Y \xrightarrow{j} X \cup_f Y$ defined by*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d] \\ Y \arrow[r, "j"'] & X \cup_f Y \end{tikzcd}$$

is a cofibration.

Proof. Observe that

$$(Y \cup_f X) \times I \cong (Y \times I) \cup_{f \times \mathrm{id}} (X \times I).$$

Indeed, given a commutative diagram

$$
\begin{tikzcd}
A \times I \arrow[r, "i\times \mathrm{id}"] \arrow[d, "f"'] & X \times I \arrow[d, "k\times\mathrm{id}"] \arrow[ddr, bend left, "F"] & \\
Y \times I \arrow[r, "j\times\mathrm{id}"'] \arrow[drr, bend right, "G"'] & X \cup_f Y \times I & \\
& & Z
\end{tikzcd}
$$

For each $t \in I$, there is a map h_t making the following diagram commute:

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d] \arrow[ddr, bend left, "F_t"] & \\
Y \arrow[r, "j"'] \arrow[drr, bend right, "G_t"'] & X \cup_f Y \arrow[dr, "h_t"] & \\
& & Z
\end{tikzcd}
$$

Then $H(p,t) = h_t(p)$ is the universal arrow.

Then using the HEP for $A \to X$ ($*$) followed by the universal property of the pushout $(Y \cup_f X) \times I$ ($\star$) , we get

$$
\begin{tikzcd}
& A \arrow[rr] \arrow[dl] \arrow[dd] & & A \times I \arrow[dl] \arrow[dd] & \\
Y \arrow[rr] \arrow[dd] & & Y \times I \arrow[dd] \arrow[dddrr] & & \\
& X \arrow[rr] \arrow[dl] & & X \times I \arrow[dl] \arrow[ddr, dotted, "*"] & \\
Y \cup_f X \arrow[rr] \arrow[drrrr, bend right] & & (Y \cup_f X) \times I \arrow[drr, dashed, "\star"] & & \\
& & & & Z
\end{tikzcd}
$$

□

Corollary 1.2. *Let* $X \xrightarrow{f} Y \xrightarrow{i} C_f$ *be a cofiber sequence. Then* $Y \xrightarrow{i} C_f$ *and* $CY \to C_i$ *are cofibrations.*

Proof. This follows from the fact that $X \to CX$ is a cofibration and that

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & CX \arrow[d] \\
Y \arrow[r] & C_f
\end{tikzcd}
$$

and

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CY \arrow[r] & C_i \end{tikzcd}$$

are pushouts. ☐

2. Quotients by contractible subspaces

Definition 2.1. A contracting homotopy is a map $H : X \times I \to X$ such that $H_0 = \mathrm{id}_X$ and $H_1 = *$.

Proposition 2.2. *Suppose that $A \subset X$ and $* \in A$. Suppose that there exists a map $H : X \times I \to X$ such that*

- $H|_{X\times\{0\}} = \mathrm{id}_X$
- $H|_{A\times I}$ *has image in A and is a contracting homotopy for A.*

Then $X \xrightarrow{q} X/A$ is a homotopy equivalence.

Proof. We need to find a map $p : X/A \to X$ and homotopies $p \circ q \simeq \mathrm{id}_X$ and $q \circ p \simeq \mathrm{id}_{X/A}$. The proof of the continuity of the maps we construct is below.

The map $q : X \to X/A$ has a set theoretic section given by

$$s(\overline{x}) = \begin{cases} x & x \notin A \\ * & x \in A. \end{cases}$$

$$\begin{tikzcd} X \arrow[r, "q"] & X/A \arrow[r, "s"] \arrow[dr, "p"'] & X \arrow[d, "H|_{X\times\{1\}}"] \\ & & X \end{tikzcd}$$

Note that $p \circ q = H|_{X\times\{1\}}$. So, H is a homotopy between id_X and $p \circ q = H|_{X\times\{1\}}$.

Define G as

$$\begin{tikzcd} X/A \times I \arrow[r, "s\times\mathrm{id}"] \arrow[drr, "G"'] & X \times I \arrow[r, "H"] & X \arrow[d, "q"] \\ & & X/A \end{tikzcd}$$

Then, $G(\overline{x}, 0) = \overline{x}$ and

$$G(\overline{x}, 1) = q \circ (H|_{X\times\{1\}} \circ s) = q \circ p.$$

So G is a homotopy between $\mathrm{id}_{X/A}$ and $q \circ p$.

Continuity of p:

For $U \subset X$ open,

$$q^{-1}(p^{-1}(U)) = (p \circ q)^{-1}(U) = (H|_{X\times\{1\}})^{-1}(U)$$

is open in X by the continuity of $H|_{X\times\{1\}}$, hence $(p^{-1}(U))$ is open in X/A and p is continuous.

Continuity of G

Note that if $U \subset X$ and $U \cap A = \emptyset$ or $A \subset U$, then $s^{-1}(U)$ is open in X/A since $q^{-1}(s^{-1}(U)) = U$ in this case.

For $\overline{U} \subset X/A$ open, let $U = q^{-1}(\overline{U})$. Then $A \subset U$ or $A \cap U = \emptyset$. Suppose that $A \subset U$. Then $A \times I \subset H^{-1}(U)$ and $(q \times \mathrm{id})^{-1}(s \times \mathrm{id})^{-1}H^{-1}(U) = H^{-1}(U)$, so that $(s \times \mathrm{id})^{-1}H^{-1}(U)$ is open. If $A \cap U = \emptyset$, then since $H_{A\times I}$ has image in A, $H^{-1}(U) \cap A \times I = \emptyset$. Again, we have $(q \times \mathrm{id})^{-1}(s \times \mathrm{id})^{-1}H^{-1}(U) = H^{-1}(U)$. □

Proposition 2.3. *Let $A \subseteq X$ subspace, with A contractible. Suppose that the inclusion $i : A \to X$ is a cofibration. Then $X \to X/A$ is a homotopy equivalence.*

Proof. Choose a contraction $h : A \times I \to A$. Composing h with the inclusion of A into X, we get a map $H : A \times X \to X$ such that the folllowing digram commtues:

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"'] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "\mathrm{id}_X"'] & X \times I \arrow[dr, dotted, "\widetilde{H}"] & \\
& & X
\end{tikzcd}$$

Since $A \to X$ is a cofibration, we can extend H to a map $\widetilde{H} : X \times I \to X$ as indicated in the diagram. Then $\widetilde{H}$ satisfies

- $\widetilde{H} : X \times \{0\} \to X$ is the identity.
- $\widetilde{H}(A \times I) = H(A \times I) = h(A \times I) \subset A$
- $\widetilde{H}(A \times \{1\}) = *$

which are the conditions of Proposition 2.2. Hence $X \to X/A$ is a homotopy equivalence. □

Example 2.4. Let $A = S^1 \backslash \{(1, 0)\}$ and consider the inclusion $A \to S^1$. Then $S^1/A \cong T$ where the topology on $T = \{a, b\}$ with open sets $\emptyset, \{a\}, \{a, b\}$. However, this is not a homotopy equivalence. In fact, T is contractible. Let $H : T \times I \to T$ be given by

$$H(x, s) = \begin{cases} a & (x, s) \neq (b, 0) \\ b & (x, s) = (b, 0). \end{cases}$$

Then $H^{-1}\{a\} = (\{a\} \times I) \cup (T \times (0,1])$ which is open, so H is continuous and gives a contraction of T onto $\{a\}$.

Definition 2.5. A based space X is well pointed if $* \to X$ is a cofibration.

Exercise 2.6. Let $SX = (X \times I)/(X \times \{0\} \cup X \times \{1\})$ and $\Sigma X = (X \times I)/(X \times \{0\} \cup X \times \{1\} \cup * \times I)$ Prove that if X is well-pointed, the natural map $SX \to \Sigma X$ is a homotopy equivalence.