

MATH 6280 - CLASS 9

CONTENTS

1. Cofibrations and the HEP - Continued	1
2. Quotients by contractible subspaces	3
3. Barratt-Puppe Sequence	5
4. Replacing $f : X \rightarrow Y$ by a cofibration	6
5. Neighborhood Deformation Retracts	7
6. Homotopy Fiber	7

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. COFIBRATIONS AND THE HEP - CONTINUED

Proposition 1.1. *The pushout of a cofibration is cofibration. That is, for $A \xrightarrow{i} X$ a cofibration and $f : A \rightarrow Y$ be a map, then $Y \xrightarrow{j} X \cup_f Y$ defined by*

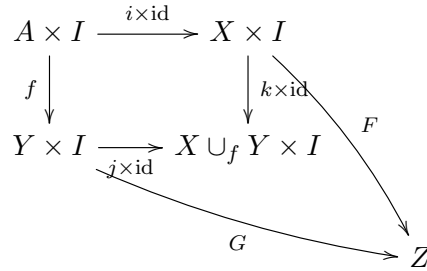
$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ f \downarrow & & \downarrow \\ Y & \xrightarrow{j} & X \cup_f Y \end{array}$$

is a cofibration.

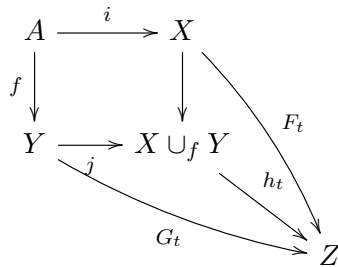
Proof. Observe that

$$(Y \cup_f X) \times I \cong (Y \times I) \cup_{f \times \text{id}} (X \times I).$$

Indeed, given a commutative diagram

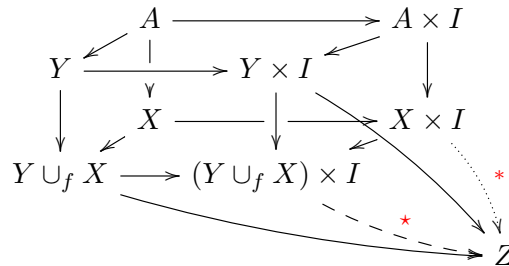


For each $t \in I$, there is a map h_t making the following diagram commute:



Then $H(p, t) = h_t(p)$ is the universal arrow.

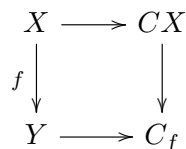
Then using the HEP for $A \rightarrow X$ (*) followed by the universal property of the pushout $(Y \cup_f X) \times I$ (*), we get



□

Corollary 1.2. Let $X \xrightarrow{f} Y \xrightarrow{i} C_f$ be a cofiber sequence. Then $Y \xrightarrow{i} C_f$ and $CY \rightarrow C_i$ are cofibrations.

Proof. This follows from the fact that $X \rightarrow CX$ is a cofibration and that



and

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CY & \longrightarrow & C_i \end{array}$$

are pushouts. □

2. QUOTIENTS BY CONTRACTIBLE SUBSPACES

Definition 2.1. A contracting homotopy is a map $H : X \times I \rightarrow X$ such that $H_0 = \text{id}_X$ and $H_1 = *$.

Proposition 2.2. Suppose that $A \subset X$ and $* \in A$. Suppose that there exists a map $H : X \times I \rightarrow X$ such that

- $H|_{X \times \{0\}} = \text{id}_X$
- $H|_{A \times I}$ has image in A and is a contracting homotopy for A .

Then $X \xrightarrow{q} X/A$ is a homotopy equivalence.

Proof. We need to find a map $p : X/A \rightarrow X$ and homotopies $p \circ q \simeq \text{id}_X$ and $q \circ p \simeq \text{id}_{X/A}$. The proof of the continuity of the maps we construct is below.

The map $q : X \rightarrow X/A$ has a set theoretic section given by

$$s(\bar{x}) = \begin{cases} x & x \notin A \\ * & x \in A. \end{cases}$$

$$\begin{array}{ccccc} X & \xrightarrow{q} & X/A & \xrightarrow{s} & X \\ & & & \searrow p & \downarrow H|_{X \times \{1\}} \\ & & & & X \end{array}$$

Note that $p \circ q = H|_{X \times \{1\}}$. So, H is a homotopy between id_X and $p \circ q = H|_{X \times \{1\}}$.

Define G as

$$\begin{array}{ccccc} X/A \times I & \xrightarrow{s \times \text{id}} & X \times I & \xrightarrow{H} & X \\ & & & \searrow G & \downarrow q \\ & & & & X/A \end{array}$$

Then, $G(\bar{x}, 0) = \bar{x}$ and

$$G(\bar{x}, 1) = q \circ (H|_{X \times \{1\}} \circ s) = q \circ p.$$

So G is a homotopy between $\text{id}_{X/A}$ and $q \circ p$.

Continuity of p :

For $U \subset X$ open,

$$q^{-1}(p^{-1}(U)) = (p \circ q)^{-1}(U) = (H|_{X \times \{1\}})^{-1}(U)$$

is open in X by the continuity of $H|_{X \times \{1\}}$, hence $(p^{-1}(U))$ is open in X/A and p is continuous.

Continuity of G

Note that if $U \subset X$ and $U \cap A = \emptyset$ or $A \subset U$, then $s^{-1}(U)$ is open in X/A since $q^{-1}(s^{-1}(U)) = U$ in this case.

For $\bar{U} \subset X/A$ open, let $U = q^{-1}(\bar{U})$. Then $A \subset U$ or $A \cap U = \emptyset$. Suppose that $A \subset U$. Then $A \times I \subset H^{-1}(U)$ and $(q \times \text{id})^{-1}(s \times \text{id})^{-1}H^{-1}(U) = H^{-1}(U)$, so that $(s \times \text{id})^{-1}H^{-1}(U)$ is open. If $A \cap U = \emptyset$, then since $H_{A \times I}$ has image in A , $H^{-1}(U) \cap A \times I = \emptyset$. Again, we have $(q \times \text{id})^{-1}(s \times \text{id})^{-1}H^{-1}(U) = H^{-1}(U)$. \square

Proposition 2.3. *Let $A \subseteq X$ subspace, with A contractible. Suppose that the inclusion $i : A \rightarrow X$ is a cofibration. Then $X \rightarrow X/A$ is a homotopy equivalence.*

Proof. Choose a contraction $h : A \times I \rightarrow A$. Composing h with the inclusion of A into X , we get a map $H : A \times X \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{i_0} & A \times I \\
 \downarrow i & & \downarrow i \times \text{id} \\
 X & \xrightarrow{i_0} & X \times I \\
 & \searrow \text{id}_X & \downarrow \tilde{H} \\
 & & X
 \end{array}$$

$\begin{array}{c} \nearrow H \\ \nearrow \tilde{H} \end{array}$

Since $A \rightarrow X$ is a cofibration, we can extend H to a map $\tilde{H} : X \times I \rightarrow X$ as indicated in the diagram. Then \tilde{H} satisfies

- $\tilde{H} : X \times \{0\} \rightarrow X$ is the identity.
- $\tilde{H}(A \times I) = H(A \times I) = h(A \times I) \subset A$
- $\tilde{H}(A \times \{1\}) = *$

which are the conditions of Proposition 2.2. Hence $X \rightarrow X/A$ is a homotopy equivalence. \square

Example 2.4. Let $A = S^1 \setminus \{(1, 0)\}$ and consider the inclusion $A \rightarrow S^1$. Then $S^1/A \cong T$ where the topology on $T = \{a, b\}$ with open sets $\emptyset, \{a\}, \{a, b\}$. However, this is not a homotopy equivalence. In fact, T is contractible. Let $H : T \times I \rightarrow T$ be given by

$$H(x, s) = \begin{cases} a & (x, s) \neq (b, 0) \\ b & (x, s) = (b, 0). \end{cases}$$

Then $H^{-1}\{a\} = (\{a\} \times I) \cup (T \times (0, 1])$ which is open, so H is continuous and gives a contraction of T onto $\{a\}$.

Definition 2.5. A based space X is well pointed if $* \rightarrow X$ is a cofibration.

Exercise 2.6. Let $SX = (X \times I)/(X \times \{0\} \cup X \times \{1\})$ and $\Sigma X = (X \times I)/(X \times \{0\} \cup X \times \{1\} \cup * \times I)$. Prove that if X is well-pointed, the natural map $SX \rightarrow \Sigma X$ is a homotopy equivalence.