MATH 6280 - CLASS 8

Contents

1. Cofibrations and the HEP 1

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Cofibrations and the HEP

Definition 1.1 (Homotopy Extension Property)**.** Let $\mathcal{C}$ be a class of topological spaces. A map $i : A \to X$ has the $\mathcal{C}$-HEP if, for every $Y \in \mathcal{C}$, the following extension problem has a solution

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times \mathrm{id}"'] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dotted, "\widetilde{H}"] & \\
 & & Y
\end{tikzcd}$$

In other words, you can extend the homotopy on $f|_A$ to one on f. This can also be phrased in terms of the following extension problem:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "p_0 = ev_0"] \\
X \arrow[r, "f"'] \arrow[ur, dotted] & Y
\end{tikzcd}$$

If $\mathcal{C}$ is the collection of all topological spaces, we say that $i : A \to X$ has the HEP and call i a *cofibration*.

Lemma 1.2. *Let $J = [0, 1]$. The following maps have the HEP:*

(a) $i_0 : X \to X \times J$

(b) $i_0 : X \to CX$

(c) Then the map $j : X \to M_f$ *which sends* x *to* $(x, 0)$ *in* M_f*, where for* $X \xrightarrow{f} Y$*,* M_f *is the pushout*

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "f"'] & X \times J \arrow[d] \\
Y \arrow[r] & M_f
\end{tikzcd} .$$

Proof.

(a-b) We need to show that there are extensions to

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \times I \arrow[d] \arrow[ddr, bend left, "H"] & \\
X \times J \arrow[r] \arrow[drr, bend right, "f"'] & (X \times J) \times I \arrow[dr, dotted] & \\
 & & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \times I \arrow[d] \arrow[ddr, bend left, "H"] & \\
CX \arrow[r] \arrow[drr, bend right, "f"'] & CX \times I \arrow[dr, dotted] & \\
 & & Y
\end{tikzcd}$$

The idea is to stretch the cylinder or the cone through the homotopy. This can be done using the homotopy

$$\widetilde{H}((x,t),s) = \begin{cases} f(x, 1-(1-t)(1+s)) & (1-t)(1+s) \leq 1 \\ H(x, (1-t)(1+s)-1) & (1-t)(1+s) \geq 1 \end{cases} \tag{1}$$

(c) For

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[d, "j"'] & X \times I \arrow[d] \arrow[ddr, bend left, "H"] & \\
M_f \arrow[r] \arrow[drr, bend right, "f"'] & M_f \times I \arrow[dr, dotted] & \\
& & Z
\end{tikzcd}$$

the extension is defined exactly as in the previous problem. We let

$$\widetilde{H}|_{(X\times I)\times I}((x,t),s) = \begin{cases} f(x, 1-(1-t)(1+s)) & (1-t)(1+s) \leq 1 \\ H(x,(1-t)(1+s)-1) & (1-t)(1+s) \geq 1 \end{cases}$$

and

$$\widetilde{H}|_{Y\times I}(y,s) = f(y,1)$$

$\square$

Remark 1.3. Let's experiment with the homotopy of the previous proof.

- $t=0$: $(1-t)(1+s) = (1+s) \geq 1$ for all s, so

$$\widetilde{H}((x,0),s) = H(x,s)$$

- $t = 1/4$: then $(1-t)(1+s) = \frac{3}{4}(1+s) \leq 1$ if and only if $s \leq \frac{1}{3}$, so

$$\widetilde{H}((x,\frac{1}{4}),s) = \begin{cases} f(x,\frac{1}{4}(1-3s)) & s \leq \frac{1}{3} \\ H(x,\frac{1}{4}(3s-1)) & s \geq \frac{1}{3} \end{cases}$$

- $t = 1/2$: then $(1-t)(1+s) = \frac{1}{2}(1+s) \leq 1$ for all s since $0 \leq s \leq 1$, so

$$\widetilde{H}((x,\frac{1}{2}),s) = f(x,\frac{1}{2}(1-s))$$

- $t = 3/4$: then $(1-t)(1+s) = \frac{1}{4}(1+s) \leq 1$ for all s, so

$$\widetilde{H}((x,\frac{3}{4}),s) = f(x,\frac{1}{4}(3-s))$$

- $t=1$: then $(1-t)(1+s) = 0 \leq 1$ for all s, so

$$\widetilde{H}((x,1),s) = f(x,1)$$

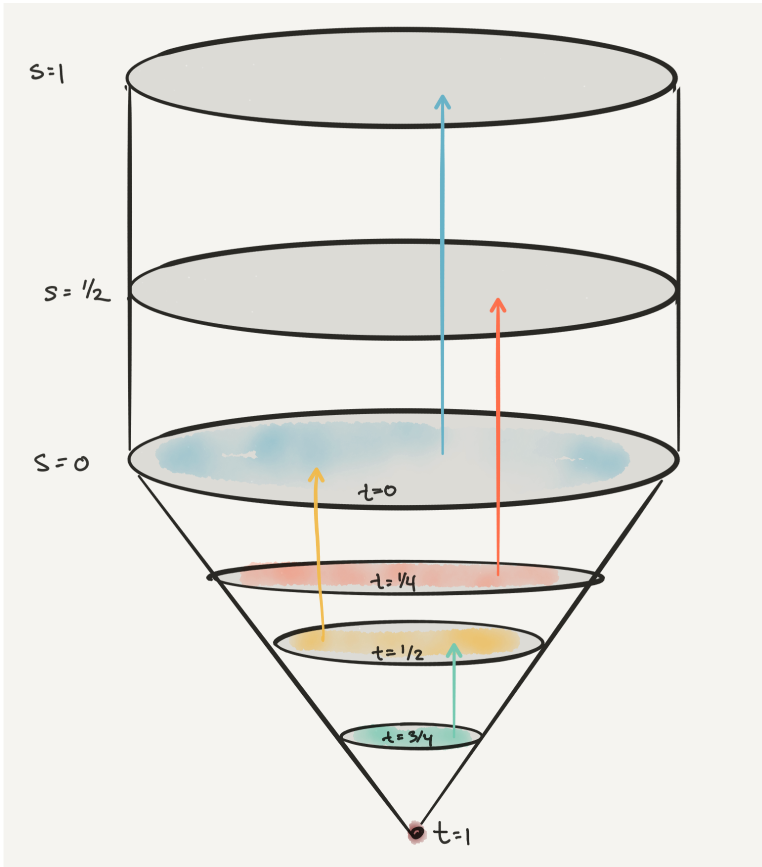

FIGURE 1. The extension $\widetilde{H}$ described in (1).

Corollary 1.4. *The inclusion $S^{n-1} \to D^n$ is a cofibration.*

Proof. Indeed, $D^{n-1} \cong CS^{n-1}$. □

Proposition 1.5 (Universal Test Diagram)**.** *Consider $A \xrightarrow{i} X$ and let M_i be the mapping cylinder defined by the pushout*

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "J"] \\ X \arrow[r, "j"'] & M_i. \end{tikzcd}$$

Then $i : A \to X$ is a cofibration if and only if there exists r making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"'] \arrow[ddr, bend left, "J"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "j"'] & X \times I \arrow[dr, dotted, "r"] & \\
& & M_i
\end{tikzcd}$$

Proof. Suppose that $i : X \to A$ is a cofibration, then this is just the HEP. Now suppose that r exists. Consider another diagram

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"'] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dotted, "\widetilde{H}?"] & \\
& & Y
\end{tikzcd}$$

Since M_i is a pushout, there is a factorization

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "i\times\mathrm{id}"'] \arrow[dd, "J"] \arrow[ddr, bend left, "H"] & \\
& X \times I \arrow[dr, "r"] & & \\
X \arrow[ur, "i_0"] \arrow[rr, "j"] \arrow[drrr, bend right, "f"'] & & M_i \arrow[dr, "H'"] & \\
& & & Y
\end{tikzcd}$$

Then $\widetilde{H} = H' \circ r$ is the desired extension. $\square$

Corollary 1.6. *If $A \subset X$, then $i : A \to X$ is a cofibration if and only if $X \times I$ is a retract of $M_i = X \times \{0\} \cup A \times I$.*

Proof. The map $r : X \times I \to M_i$ has the property that $M_i \to X \times I \to M_i$ is the identity, so it is a retract. $\square$

Corollary 1.7. *A cofibration $i : A \to X$ is an injection. Further, if X is Hausdorff, $i(A)$ is closed in X.*

Proof. Let J and and r be as above. Then

$$J(a,1) = r(i(a),1).$$

Since $J|_{A\times\{1\}}$ is the identity on A, this implies that $i(a) \neq i(a')$ if $a \neq a'$.

Since $A \to X$ is a cofibration, so is $i(A) \to X$. Hence, $X \times I$ retracts onto $X \times \{0\} \cup i(A) \times I$. For a Hausdorff space, the image of a retract is closed, so $X \times \{0\} \cup i(A) \times I$ is a closed subspace of $X \times I$. Therefore, intersecting with $X \times \{1\}$, we conclude that $A \times \{1\}$ is closed in $X \times \{1\}$ so that A is closed in X. $\square$

Exercise 1.8. Let $X = \{0,1\}$ with the trivial topology. Let $A = \{0\}$. Show that the inclusion $A \to X$ is a cofibration whose image is not closed.

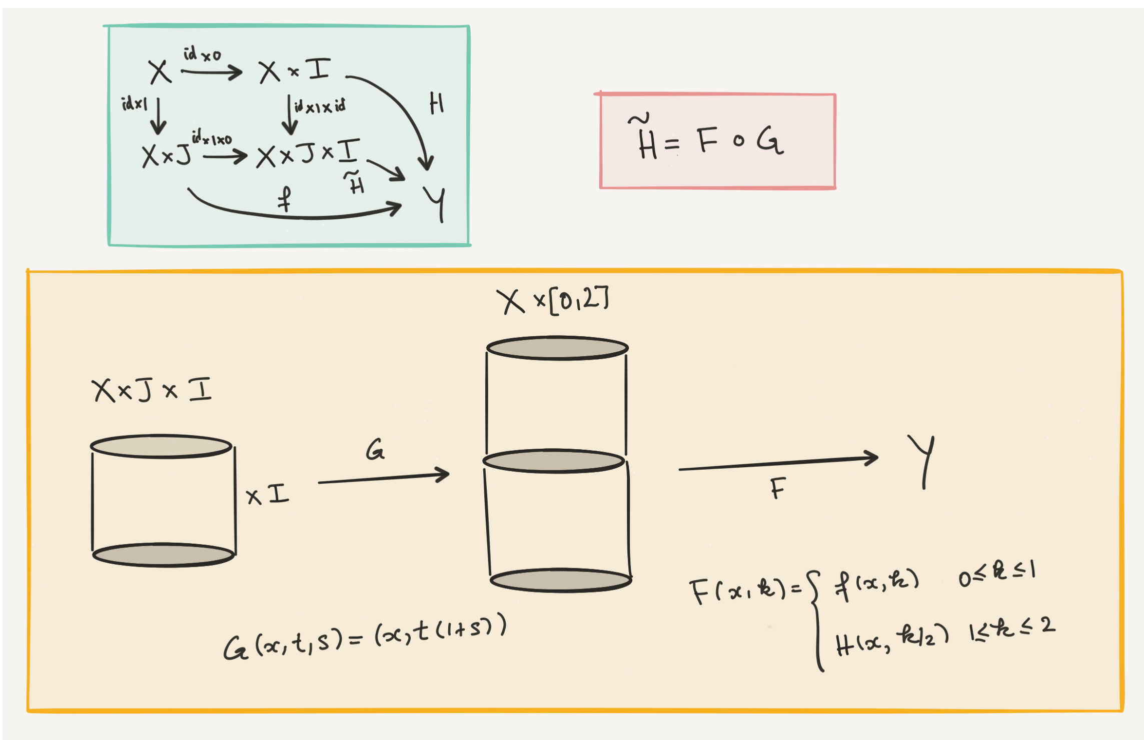

FIGURE 2. Another depiction of (1).