MATH 6280 - CLASS 7

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

Proposition 0.1. *If Q is an H–cogroup and W is an H–group, then the two group structures on $[Q, W]_*$ are equal and this is in fact an abelian group.*

Proof. Let $[a]$, $[b]$, $[c]$ and $[d]$ be elements of $[Q, W]_*$ with representatives a, b, c, d. Let

$$[a] * [b] = [\mu \circ (a \times b)]$$

that is, the composite

$$Q \xrightarrow{a\times b} W \times W \xrightarrow{\mu} W \ .$$

Let

$$[a] \otimes [b] = [(a \vee b) \circ \nu]$$

that is, the composite

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{a\vee b} W \ .$$

We must show that

$$([a] * [c]) \otimes ([b] * [d]) = ([a] \otimes [b]) * ([c] \otimes [d]).$$

It's enough to show that

$$(\mu \circ (a \times b) \vee \mu \circ (c \times d)) \circ \nu = \mu \circ ((a \vee c) \circ \nu \times (b \vee d) \circ \nu)$$

However,

$$(\mu \circ (a \times b) \vee \mu \circ (c \times d)) \circ \nu = \mu \circ ((a \times b) \vee (c \times d)) \circ \nu$$

and

$$\mu \circ ((a \vee c) \circ \nu \times (b \vee d) \circ \nu) = \mu \circ ((a \vee c) \times (b \vee d)) \circ \nu.$$

However, the maps

$$(a \times b) \vee (c \times d) : Q \vee Q \to W \times W$$

and

$$(a \vee c) \times (b \vee d) : Q \vee Q \to W \times W$$

are equal (this is easy to check on elements). □

1. Homotopy cofiber

In the category of abelian groups, one can take kernels and cokernels. They satisfy certain universal properties:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r] \arrow[d, "g"] & \operatorname{coker}(f) \arrow[dl, dotted] \\ & C & \end{tikzcd}$$

and

$$\begin{tikzcd} & C \arrow[dl, dotted] \arrow[d, "g"'] \arrow[dr, "0"] & \\ \ker(f) \arrow[r] & A \arrow[r, "f"'] & B \end{tikzcd}$$

There are analogous constructions in the homotopy category of topological spaces, where a map being zero is replaced by a map being null-homotopic. These are called the homotopy cofibers and fibers.

Construction.
- *Let $A \subset X$ be a subspace (which contains the base point if we are in a based setting). Then given a map $f : A \to Y$,*

$$X \cup_f Y = (X \sqcup Y)/(a \sim f(a))$$

is the pushout

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d] \\ Y \arrow[r] & X \cup_f Y. \end{tikzcd}$$

- *The* mapping cylinder M_f *of* $f : X \to Y$ *is the pushout*

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d, "f"'] & X \times I \arrow[d] \\ Y \arrow[r] & M_f = Y \cup_f (X \times I) \end{tikzcd}$$

- *The* cone *on* X *is the pushout*

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d] & X \times I \arrow[d] \\ * \arrow[r] & CX \end{tikzcd}$$

- *The* mapping cone C_f *of* f, *or* homotopy cofiber of f

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d, "f"'] & CX \arrow[d] \\ Y \arrow[r] & C_f = Y \cup_f CX \end{tikzcd} \qquad \textit{or} \qquad \begin{tikzcd} X \arrow[r, "i_1"] \arrow[d] & M_f \arrow[d] \\ * \arrow[r] & C_f \end{tikzcd}$$

- $SX \cong C_f/Y$ *where* SX *is the pushout*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & SX \end{tikzcd}$$

Lemma 1.1. *A map* $f : X \to Y$ *is null-homotopic if and only if there exists a map* $F : CX \to Y$ *extending* f*:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & Y \\ CX \arrow[ur, "F"'] & \end{tikzcd}$$

Proof. A null-homotopy for f is a map $H : X \times I \to Y$ such that $H|_{X\times\{1\}}$ is the constant map and $H|_{X\times\{0\}} = f$. In other words, it is equivalent to a diagram:

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d] & X \times I \arrow[d] \arrow[ddr, "H"] & X \arrow[l, "i_0"'] \arrow[dd, "f"] \\
* \arrow[r] \arrow[drr] & CX \arrow[dr, dotted] & \\
 & & Y
\end{tikzcd}$$

Since the left square is a pushout, we see that H exists if and only if f extends to CX. $\square$

Example 1.2. $f : S^n \to X$ is trivial if and only if it extends to D^{n+1} since D^{n+1} is homeomorphic to CS^n.

Proposition 1.3. *Consider maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. The composite $g \circ f$ is null homotopic if and only if g extends to Cf:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "*"'] & Y \arrow[r] \arrow[d, "g"] & C_f \arrow[dl, dotted] \\
 & Z &
\end{tikzcd}$$

Proof. The following diagram shows that an extension of g to Cf is equivalent to an extension of $g \circ f$ to to CX, which is equivalent to $g \circ f$ being null-homotopic.

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[d, "f"'] & CX \arrow[d] \arrow[ddr] & \\
Y \arrow[r] \arrow[drr, "g"'] & C_f \arrow[dr, dotted] & \\
 & & Z
\end{tikzcd}$$

$\square$

Exercise 1.4. Let Z be path connected. Suppose that $\pi_{n-1}Z = 0$. For $g : Y \to Z$ and $S^{n-1} \to Y$, g extends to a map $Y \cup_f D^n$.

Definition 1.5. The sequence $X \xrightarrow{f} Y \to C_f$ is a called a *homotopy cofiber sequence*.

Goal (Barratt-Puppe Sequence)**.** We now have a sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \to C_f/Y \cong SX.$$

The goal is to show that

$$SX \cong C_f/Y \cong C_i/CY \simeq C_i$$

and construct a sequence

$$X \xrightarrow{f} Y \to C_f \to SX \xrightarrow{\Sigma f} SY \to C_{Sf} \to S^2X \to \dots$$

in which all triples are, up to homotopy equivalence, homotopy cofiber sequences.

In order to achieve the goal, we need to understand the homotopy type of certain quotients.