MATH 6280 - CLASS 6

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. H–cospaces

Definition 1.1. A pointed topological space Q is an H–cospace if it has a map

$$\nu : Q \to Q \vee Q$$

for which $e : Q \to Q$, $e(q) = *$ is a counit up to homotopy. That is

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(\mathrm{id},e)} Q$$

and

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(e,\mathrm{id})} Q$$

are homotopic to the identity.

There is a corresponding notion of homotopy co-associative, and of homotopy co-inverses $\tau : Q \to Q$, of H–cogroup, and homotopy co-commutative H–cogroup. A co-group homomorphism is a continuous map $k : Q \to Q'$ which makes the following diagram commute:

$$\begin{tikzcd} Q \arrow[r, "\nu'"] \arrow[d, "k"] & Q \vee Q \arrow[d, "k \vee k"] \\ Q' \arrow[r, "\nu"] & Q \vee Q \end{tikzcd}$$

Theorem 1.2. $[Q,-]_* : \mathrm{Top}_* \to \mathrm{Gr}$ *a functor to* Gr *if and only if* Q *is an* H*–cogroup. Further, if* $k : Q \to Q'$ *is a homomorphism of* H*–cogroups, then* $[Q',]_* \to [Q,]_*$ *is a natural transformation.*

Proof. If Q is an H-cogroup, then

$$\nu : Q \to Q \vee Q$$

induces maps

$$[Q \vee Q, X]_* \to [Q, X]_*$$

However, $[Q \vee Q, X]_* \cong [Q, X]_* \times [Q, X]_*$. This gives the group multiplication.

If $[Q,-]_*$ is a functor to groups, letting $i_1, i_2 : Q \to Q \vee Q$ be the inclusions at the base point, the comultiplication ν is any map homotopic to $[i_1][i_2]$ in $[Q, Q \vee Q]_*$. □

Example 1.3. Let $\nu : \Sigma X \to \Sigma X \vee \Sigma X$ be defined by the pinch map:

$$\nu(x \wedge t) = \begin{cases} (x \wedge 2t, *) & 0 \le t \le 1/2 \\ (*, x \wedge (2t-1)) & 1/2 \le t \le 1. \end{cases}$$

The inverse can be defined by

$$\tau(x \wedge t) = x \wedge (1-t).$$

Given any map $f : X \to Y$, we get a group homomorphism

$$\Sigma f : \Sigma X \to \Sigma Y$$

defined by

$$\Sigma(f)(x \wedge t) = f(x) \wedge t.$$

Recall:

Proposition 1.4. *There is a natural homeomorphism*

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y)$$

such that

$$f \mapsto \widetilde{f}(x)(t) = f(x \wedge t)$$

Exercise 1.5. Check that this induces a group isomorphism

$$[\Sigma X, Y]_* \to [X, \Omega Y]_*$$

which is natural in both X and Y.

Lemma 1.6 (Eckman-Hilton argument)**.** *Let X be a set and let $*, \otimes : X \times X \to X$ be two unital binary operations with the same unit $e \in X$. Suppose that*

$$(a \otimes b) * (c \otimes d) = (a * c) \otimes (b * d)$$

Then $* = \otimes$ and the operation is both commutative and associative.

Proof. Exercise. $\square$

Proposition 1.7. *If Q is an H–cogroup and W is an H–group, then the two group structures on $[Q, W]_*$ are equal and this is in fact an abelian group.*

Proof. Let $[a]$, $[b]$, $[c]$ and $[d]$ be elements of $[Q, W]_*$ with representatives a, b, c, d. Let

$$[a] * [b] = [\mu \circ (a \times b)]$$

that is, the composite

$$Q \xrightarrow{a \times b} W \times W \xrightarrow{\mu} W \ .$$

Let

$$[a] \otimes [b] = [(a \vee b) \circ \nu]$$

that is, the composite

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{a \vee b} W \ .$$

We must show that

$$([a] \otimes [b]) * ([c] \otimes [d]) = ([a] * [c]) \otimes ([b] * [d]).$$

It's enough to show that

$$(\mu \circ (a \times b) \vee \mu \circ (c \times d)) \circ \nu = \mu \circ ((a \vee c) \circ \nu \times (b \vee d) \circ \nu)$$

However,

$$(\mu \circ (a \times b) \vee \mu \circ (c \times d)) \circ \nu = \mu \circ ((a \times b) \vee (c \times d)) \circ \nu$$

and

$$\mu \circ ((a \vee c) \circ \nu \times (b \vee d) \circ \nu) = \mu \circ ((a \vee c) \times (b \vee d)) \circ \nu.$$

However, the maps

$$(a \times b) \vee (c \times d) : Q \vee Q \to W \times W$$

and

$$(a \vee c) \times (b \vee d) : Q \vee Q \to W \times W$$

are equal (this is easy to check on elements). $\square$

2. Higher Homotopy Groups

Exercise 2.1.
- There are homeomorphisms $S^n \cong S^1 \wedge S^{n-1}$

Definition 2.2. The n'th homotopy group of X is

$$\begin{aligned}\pi_n X &= [S^n, X]_* \\ &\cong [S^{n-1}, \Omega X]_* \\ &\cong \dots, \\ &\cong [S^0, \Omega^n X]_*\end{aligned}$$

where $\Omega^n X = \underbrace{\Omega\Omega\dots\Omega X}_{n}$.

Corollary 2.3. *If $n = 1$, $\pi_1 X$ is a group. If $n \geq 2$, then $\pi_n X$ is an abelian group.*

Exercise 2.4. Let X_α be a collection of based path connected spaces. Then $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n X_\alpha$.

Definition 2.5.
- A space X is n–connected if $\pi_k X = 0$ for $k \leq n$.
- A map $f : X \to Y$ is n–connected or an n–equivalence if $\pi_k f$ is an isomorphism for $k < n$ and onto for $k = n$.

3. Homotopy cofiber

In the category of abelian groups, one can take kernels and cokernels. They satisfy certain universal properties:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r] \arrow[d, "g"] & \mathrm{coker}(f) \arrow[dl, dotted] \\ & C & \end{tikzcd}$$

and

$$\begin{tikzcd} & C \arrow[dl, dotted] \arrow[d, "g"'] \arrow[dr, "0"] & \\ \ker(f) \arrow[r] & A \arrow[r, "f"'] & B \end{tikzcd}$$

There are analogous constructions in the homotopy category of topological spaces, where a map being zero is replaced by a map being null-homotopic. These are called the homotopy cofibers and fibers.