MATH 6280 - CLASS 5

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

Notation.

- $\mathrm{Map}(X, Y)$: continuous functions from X to Y with the compact-open topology
- $[X, Y]$: homotopy classes of unbased maps from X to Y.
- $\mathrm{Map}_*(X, Y)$: continuous base point preserving functions from X to Y with the subspace topology
- $[X, Y]_*$: based homotopy classes of based maps from X to Y
- $PX = \mathrm{Map}_*(I, X)$: paths in X starting at the base-point.
- $\Omega X = \mathrm{Map}_*(S^1, X)$: based loops in X.
- $X \wedge Y = (X \times Y)/(* \times Y \cup X \times *)$: the smash product of X and Y
- $\Sigma X = X \wedge S^1$: the reduced suspension of X
- $SX = X \times I/(X \times \{0, 1\})$: the unreduced suspension of X

Remark 0.1. We will assume that the spaces X and Y are nice enough. This can mean locally compact and Hausdorff, or a more general notion of compactly generated. For such spaces, we have

$$\pi_0 \,\mathrm{Map}(X, Y) = [X, Y]$$

and

$$\pi_0 \,\mathrm{Map}_*(X, Y) = [X, Y]_*$$

1. H–Spaces

Recall that a topological group is a topological space G with a continuous multiplication

$$\mu : G \times G \to G$$

giving G the structure of a group, such that the map

$$i : G \to G$$

sending $x \to x^{-1}$ is continuous.

Example 1.1.
- S^1 viewed as the units in the complex numbers
- $\mathbb{R}$ with addition
- $\mathbb{R}^*$ with multiplication
- Various matrix groups, $GL_n(\mathbb{R})$, $SO(n)$, etc.

Exercise 1.2. If G is a topological group with base point the identity e, then the map

$$\mathrm{Map}_*(X, G) \times \mathrm{Map}_*(X, G) \to \mathrm{Map}_*(X, G)$$

which sends f and g to

$$(fg)(x) = f(x)g(x)$$

makes $\mathrm{Map}_*(X, G)$ into a group, which is abelian if G is abelian. The unit is the constant map at the identity of G.

Further, the mapping into G gives a functor

$$\mathrm{Map}_*(-, G) : \mathrm{Top}_* \to \mathrm{TopGr}_*.$$

That is,

- If $X \to Y$ is continuous, then $\mathrm{Map}_*(Y, G) \to \mathrm{Map}_*(X, G)$ is a group homomorphism.

Further,

- If $g : G \to H$ is a continuous homomorphism of topological groups, then $\mathrm{Map}_*(X, G) \to \mathrm{Map}_*(X, H)$ is a group homomorphism and this is natural in X. That is, $\mathrm{Map}_*(-, g)$ is a natural transformation.

Similarly, this induces a group structure on $[X, G]_*$ with the same properties.

Question 1.3. For what space W is

$$[-, W]_* : \mathrm{Top}_* \to \mathrm{Gr}?$$

Note that if $[-, W]_*$ is a functor to groups, then $[*, W]_*$ maps to the group with one element. The map $X \to *$ then induces a group homomorphism $[*, W]_* \to [X, W]_*$ which necessarily picks out the unit. Therefore,

(1) The constant map at the base point $e : X \to W$, $e(x) = *$ is the identity.
(2) Every based continuous maps $X \to Y$ induce group homomorphisms $f^* : [Y, W]_* \to [X, W]_*$.

Similarly, when is $[Q, -]_*$ a functor to Gr?

Definition 1.4. An H-space W is a pointed topological space and a continuous map

$$\mu : W \times W \to W$$

such that the map $e : W \to W$, $e(w) = *$ is a unit up to homotopy. That is

$$W \xrightarrow{(e,\mathrm{id})} W \times W \xrightarrow{\mu} W$$

and

$$W \xrightarrow{(\mathrm{id},e)} W \times W \xrightarrow{\mu} W$$

are homotopic to the identity.

(1) W is homotopy associative if the following diagram commutes up to homotopy

$$\begin{tikzcd} W \times W \times W \arrow[r, "(\mu,\mathrm{id})"] \arrow[d, "(\mathrm{id},\mu)"] & W \times W \arrow[d, "\mu"] \\ W \times W \arrow[r, "\mu"] & W. \end{tikzcd}$$

(2) A map $j : W \to W$ gives inverses to W up to homotopy if

$$W \xrightarrow{(\mathrm{id},j)} W \times W \xrightarrow{\mu} W$$

and

$$W \xrightarrow{(j,\mathrm{id})} W \times W \xrightarrow{\mu} W$$

are null-homotopic.

(3) W is homotopy commutative if the following diagram commutes up to homotopy

$$\begin{tikzcd} W \times W \arrow[r, "s"] \arrow[dr, "\mu"'] & W \times W \arrow[d, "\mu"] \\ & W \end{tikzcd}$$

where $s(w_1, w_2) = (w_2, w_1)$.

(4) If W satisfies (1) and (2), it is called an H-group and it is a homotopy abelian if it satisfies (3).

(5) A homomorphism of H–spaces is a based map $h : W \to W'$ which makes the following diagram commute up to homotopy:

$$\begin{tikzcd} W \times W \arrow[r, "\mu"] \arrow[d, "h"] & W \arrow[d, "h"] \\ W' \times W' \arrow[r, "\mu"] & W' \end{tikzcd}$$

Example 1.5.
- Any topological group is an H–space.
- $\mathbb{C}P^\infty$ is an H–space. Recall that $\mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n$. Let $x = [a_0 : a_1 : \ldots : a_n] \in \mathbb{C}P^n$ and $y = [b_0 : b_1 : \ldots : b_m] \in \mathbb{C}P^m$ be two points. The coefficients of

$$(a_0 + \ldots + a_n z^n)(b_0 + \ldots + b_m z^m)$$

determine a point $\mu(x, y) \in \mathbb{C}P^{m+n}$. The map μ gives $\mathbb{C}P^\infty$ the structure of an H–space. Here is an interesting post by John Baez on this topic.

Some answer to a question: Although this description of the multiplication on $\mathbb{C}P^\infty$ does not have inverses, $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ and up to homotopy, $K(\mathbb{Z}, 2)$ has a unique H–space structures. Indeed, $K(\mathbb{Z}, 2) \simeq K(\mathbb{Z}, 2)$ and, up to homotopy, maps $K(\mathbb{Z}\times\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$ are determined by group homomorphisms $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$. Asking that $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$ be unital up to homotopy means that it is induced by the addition $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. In fact, this gives $K(\mathbb{Z}, 2)$ the structure of an H–group. Also, $\mathbb{C}P^\infty \simeq BU(1)$ and, for an abelian group G, BG is a topological group. Therefore, $\mathbb{C}P^\infty$ is homotopy equivalent to a topological group.

- Let X be a based space. Then ΩX is an H–group with multiplication

$$\Omega X \times \Omega X \xrightarrow{\mu} \Omega X$$

given by

$$\mu(\alpha, \beta)(t) = \begin{cases} \alpha(2t) & 0 \le t \le 1/2 \\ \beta(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Let

$$j : \Omega X \to \Omega X$$

be defined by

$$j(\alpha)(t) = \alpha(1-t).$$

This makes ΩX into an H–group: the proof is very similar to that of checking that $\pi_1 X$ is a group.

Let $g : X \to Y$ be a map of based topological spaces. Then

$$\Omega g : \Omega X \to \Omega Y$$

is defined by $\alpha \mapsto g \circ \alpha$ is a homomorphism of H–groups.

In fact, $\mathbb{C}P^\infty$ itself is a loop space.

- The James construction on a based space X is an H–space:

$$J(X) = \coprod_{k \geq 0} X^k / (x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, x_k) \sim (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$$

with

$$\mu((x_1, \dots, x_k), (y_1, \dots, y_{k'})) = (x_1, \dots, x_k, y_1, \dots, y_{k'}).$$

For X a connected CW-complex, $J(X) \simeq \Omega \Sigma X$.

- S^7 as the units in the octonions $\mathbb{O}$ is an H-space but not an H–group.

Here is a remarkable result:

Theorem 1.6 (John Hubbuck, *On homotopy commutative H–spaces*)**.** *Let X be a non contractible, connected,finite complex which is a homotopy commutative H-space, then X has the homotopy type of a torus.*

Theorem 1.7. *$[-, W]_*$ is a functor to the category of groups in the sense of Question 1.3 if and only if W is an H-group. If W is homotopy abelian, then $[-, W]_*$ is a functor to abelian groups. Further, if $h : W \to W'$ is a homomorphism of H–groups, then $[-, W]_* \to [-, W']_*$ is a natural transformation.*

Proof. Suppose that $[X, W]_*$ is a group for every X. The projections $p_1, p_2 : W \times W \to W$ represent classes $[p_1]$ and $[p_2]$ in $[W \times W, W]_*$. Let μ be any map that represents the homotopy class of the product $[p_1][p_2] \in [W \times W, W]_*$. Let i represent $[\mathrm{id}]^{-1} \in [W, W]_*$. These maps will give W the structure of an H–group.

We check associativity and leave the other verifications as exercises. Let $p_1, p_2 : W \times W \to W$ and $q_1, q_2, q_3 : W \times W \times W \to W$ and let $i_1, i_2 : W \to W \times W$ be the inclusions $i_1(w) = (w, *)$ and $i_2(w) = (*, w)$. Consider,

$$f(w_1, w_2, w_3) = \mu(\mu(w_1, w_2), w_3).$$

Then

$$\begin{aligned} f(w_1, w_2, w_3) &\simeq [p_1 \circ (\mu \times \mathrm{id})][p_2 \circ (\mu \times \mathrm{id})] \\ &\simeq [p_1 \circ (\mu \times \mathrm{id})][q_3] \end{aligned}$$

However, $p_1 \circ (\mu \times \mathrm{id}) : W \times W \times W \to W \times W$ is homotopic to

$$W \times W \times W \xrightarrow{q_1 \times q_2 \times \mathrm{id}} W \times W \times W \xrightarrow{\mu \times \mathrm{id}} W \times W \xrightarrow{p_1} W$$

which is equal to

$$W \times W \times W \xrightarrow{q_1 \times q_2} W \times W \xrightarrow{\mu} W$$

So

$$\begin{aligned} f(w_1, w_2, w_3) &\simeq [p_1 \circ (\mu \times \mathrm{id})][q_3] \\ &\simeq ([\mu \circ (q_1 \times q_2)])[q_3] \\ &\simeq ([p_1 \circ (q_1 \times q_2)][p_2 \circ (q_1 \times q_2)])[q_3] \\ &\simeq ([q_1][q_2])[q_3]. \end{aligned}$$

Now, we use the associativity of the group operation and reverse engineer. □

2. H–cospaces

Definition 2.1. A pointed topological space Q is an H–cospace if it has a map

$$\nu : Q \to Q \vee Q$$

for which $e : Q \to Q$, $e(q) = *$ is a counit up to homotopy. That is

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(\mathrm{id}, e)} Q$$

and

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(e, \mathrm{id})} Q$$

are homotopic to the identity.

There is a corresponding notion of homotopy co-associative, and of homotopy co-inverses $\tau : Q \to Q$, of H–cogroup, and homotopy co-commutative H–cogroup. A co-group homomorphism is a continuous map $k : Q \to Q'$ which makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[r, "\nu'"] \arrow[d, "k"] & Q \vee Q \arrow[d, "k \vee k"] \\
Q' \arrow[r, "\nu"] & Q \vee Q
\end{tikzcd}$$

Theorem 2.2. $[Q, -]_* : \mathrm{Top}_* \to \mathrm{Gr}$ *a functor to* Gr *if and only if* Q *is an* H*–cogroup. Further, if* $k : Q \to Q'$ *is a homomorphism of* H*–cogroups, then* $[Q',]_* \to [Q,]_*$ *is a natural transformation.*

Proof. If Q is an H-cogroup, then

$$\nu : Q \to Q \vee Q$$

induces maps

$$[Q \vee Q, X]_* \to [Q, X]_*$$

However, $[Q \vee Q, X]_* \cong [Q, X]_* \times [Q, X]_*$. This gives the group multiplication.

If $[Q, -]_*$ is a functor to groups, letting $i_1, i_2 : Q \to Q \vee Q$ be the inclusions at the base point, the comultiplication ν is any map homotopic to $[i_1][i_2]$ in $[Q, Q \vee Q]_*$. $\square$