MATH 6280 - CLASS 36

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Spectra

Recall that a *Brown functor* is a functor $T : \mathrm{hCW}_*^{op} \to \mathrm{Sets}_*$ which satisfies the additivity axiom and Mayer-Vietoris (i.e., for an excisive triad $(X; A, B)$, the natural map

$$T(X) \to T(A) \times_{T(A\cap B)} T(B)$$

is surjective). Last time, Sebastian and Cherry sketched the proof of the following theorem:

Theorem 1.1 (Brown Representability)**.** *Every Brown functor is representable by a path-connected based CW-complex Y. That is, for any $X \in \mathrm{hCW}_*$*

$$T(X) = [X, Y]_*.$$

Let $\widetilde{E}^* : \mathrm{CW}_*^{op} \to \mathrm{Ab}$ be a generalized reduced cohomology theory. We can include $\mathrm{Ab} \to \mathrm{Sets}_*$ by sending A to the underlying set pointed at 0. Then, for each n, $\widetilde{E}^n$ can be viewed as a functor:

$$\widetilde{E}^n : \mathrm{hCW}_*^{op} \to \mathrm{Sets}_*$$

and by the Mayer-Vietoris theorem and the additivity axiom, these are Brown-functors. Therefore, each E_n is representable! That is, there exists pointed CW–complexes E_n such that

$$\widetilde{E}^n(X) = [X, E_n]_*.$$

Further, by the Yoneda lemma, given any two such CW-complexes E_n and E_n', there there exists an isomorphism $\phi : E_n \to E_n'$ in hCW_*, i.e., a homotopy equivalence. Hence, the sequence E_n is determined uniquely up to homotopy equivalences.

Theorem 1.2 (Milnor)**.** *If X is a CW-complex, then ΩX has the homotopy type of a CW complex.*

Note that here, there is no mention of the Suspension isomorphisms. Therefore, one expects that this will correspond to extra data. Indeed, the suspension isomorphisms give a natural transformation between the functors

$$[-, E_n]_* \to [\Sigma(-), E_{n+1}]_*.$$

Further, $[\Sigma(-), E_{n+1}]_*$ is naturally isomorphic to $[-, \Omega E_{n+1}]_*$. Let $Y \xrightarrow{\gamma} \Omega E_{n+1}$ be a CW–approximation. Then there is a homotopy equivalence $E_n \to Y$, which we can compose with γ to obtain a weak homotopy equivalence:

$$\sigma_n : E_n \to \Omega E_{n+1}.$$

Definition 1.3. A sequence of spaces $\{E_n\}_{n\geq 0}$ and together with the data of weak homotopy equivalences $\omega_n : E_n \to \Omega E_{n+1}$ is called a Ω-*prespectrum*. We often abbreviate this data by E.

Theorem 1.4 (Brown-Representability)**.** *If $\widetilde{E}$ is a generalized reduced cohomology theory, then there exists an Ω–prespectrum E which represents $\widetilde{E}$.*

We also have:

Theorem 1.5. *Any Ω–prespectrum E represents a reduced cohomology theory $\widetilde{E} : \mathrm{hCW}_*^{op} \to \mathrm{Ab}$ by*

$$\widetilde{E}^n(X) = \begin{cases} [X, E_n]_* & n \geq 0 \\ [X, \Omega^{-n} E_0]_* & n \leq 0. \end{cases}$$

Proof. Note that the additivity axiom is immediate and, given (X, A) a CW–pair, we have that $A \xrightarrow{i} X$ is a cofibration and hence, $C_i \to X/A$ is a homotopy equivalence and

$$[X/A, E_n]_* \xrightarrow{\cong} [C_i, E_n]_* \to [X, E_n]_* \to [A, E_n]_*$$

is exact. The suspension isomorphism are given by the composites:

$$[-, E_n] \xrightarrow{\omega_{n*}} [-, \Omega E_{n+1}]_* \to [\Sigma(-), E_{n+1}]_*.$$

Finally, note that the values are indeed abelian groups since

$$\widetilde{E}^n(X) \cong \widetilde{E}^{n+1}(\Sigma X) \cong \widetilde{E}^{n+2}(\Sigma^2 X) = [\Sigma^2 X, E_{n+2}]_* \cong [\Sigma X, \Omega E_{n+2}]_*,$$

which is an abelian group. □

Example 1.6. (1) Let A be an abelian group. Recall that $\Omega K(A, n+1)$ has the homotopy type of a $K(A, n)$. Indeed,

$$\pi_q \Omega K(A, n+1) = \pi_{q+1} K(A, n+1)$$

for all $q \geq 0$. Let $HA_n = K(A, n)$ if $n \geq 1$ and $HA_0 = A$ viewed as a discrete set. Fix homotopy equivalences $\omega_n : K(A, n) \to \Omega K(A, n+1)$. Let $HA = \{K(A, n), \omega_n\}$. Then $HA^n(X) = [X, HA_n]_*$ is a cohomology theory. Further, it satisfies the dimension axiom since $HA^n(S^0) = \pi_0 K(A, n)$ which is A if $n = 0$ and 0 otherwise. Hence, HA represents singular homology $H^n(-; A)$.

(2) *Bott Periodicity* implies that the sequence

$$(BU \times \mathbb{Z}, U, BU \times \mathbb{Z}, U, \ldots)$$

is a spectrum called the K, the K-theory spectrum.

(3) *Real Bott Periodicity* implies that the sequence

$$(BO \times \mathbb{Z}, O, O/U, U/Sp, \mathbb{Z} \times BSp, Sp, Sp/U, U/O, BO \times \mathbb{Z}, \ldots)$$

called the real K–theory spectrum and denoted KO.

Theorem 1.7. *Let $\widetilde{E} : \mathrm{CW}_* \to \mathrm{Ab}$ and $\widetilde{F} : \mathrm{CW}_* \to \mathrm{Ab}$ be generalized cohomology theories and suppose that $\phi : \widetilde{E} \to \widetilde{F}$ is a map of cohomology theories such that*

$$\phi : \widetilde{E}^*(S^0) \to \widetilde{F}^*(S^0)$$

is an isomorphism. Then $\phi : \widetilde{E}^*(X) \to \widetilde{F}^*(X)$ is an isomorphism for all $X \in \mathrm{CW}_*$.

Proof. First, note that since the following diagram commutes,

$$\begin{tikzcd}
\widetilde{E}^{n-q}(S^0) \arrow[r, "\phi_{n-q}"] \arrow[d, "\Sigma^q"'] & \widetilde{F}^{n-q}(S^0) \arrow[d, "\Sigma^q"] \\
\widetilde{E}^{n}(S^q) \arrow[r, "\phi_n"] & \widetilde{F}^{n}(S^q)
\end{tikzcd}$$

we have that $\phi : \widetilde{E}^*(S^q) \to \widetilde{F}^*(S^q)$ is an isomorphism for all $q \geq 0$. Let $\widetilde{E}$ and $\widetilde{F}$ be represented by Ω-prespectra E and F whose spaces are CW-complexes. Since $\phi_n : \widetilde{E}^n \to \widetilde{F}^n$ are natural transformations, there are maps

$$E_n \xrightarrow{\rho_n} F_n$$

which induce ϕ_n as

$$[-, E_n] \xrightarrow{\rho_{n*}} [-, F_n].$$

Further,

$$\pi_q(E_n) = [S^q, E_n]_* \xrightarrow{\rho_{n*}} [S^q, F_n]_* = \pi_q(F_n)$$

are isomorphisms for all $q \geq 0$ since $\widetilde{E}$ and $\widetilde{F}$ agree on spheres. That is, ρ_n are weak equivalences. The Whitehead theorem implies that for any CW–complex X,

$$[X, E_n]_* \xrightarrow{\rho_{n*}} [X, F_n]_*$$

is an isomorphism. So, the two cohomology theories agree. $\square$