MATH 6280 - CLASS 4

Contents

Example 0.1. Consider the natural numbers as a poset as follows: $a \preceq b$ if $a|b$. We can turn this into a category by letting the objects be natural numbers and the $\mathrm{Hom}(a, b)$ be nonempty if $a \preceq b$ and empty otherwise. Then the $a \times b = \gcd(a, b)$ and $a \sqcup b = \mathrm{lcm}(a, b)$.

1. Limits and Colimits continued

Example 1.1 (Limit: inverse limit)**.** Consider a diagram

$$\ldots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0$$

The inverse limit $\lim_n = \varprojlim_n X_n$ is an object in $\mathcal{C}$ with compatible maps $\varprojlim_n X_n \to X_i$ for each i and

$$\begin{tikzcd}
Z \arrow[dr, dotted] \arrow[drrr] \arrow[drrrr] \arrow[drrrrrr] \arrow[drrrrrrr] & & & & & & & \\
& \varprojlim_n X_n \arrow[r] & \ldots \arrow[r] & X_{n+1} \arrow[r, "i_{n+1}"'] & X_n \arrow[r] & \ldots \arrow[r, "i_2"'] & X_1 \arrow[r, "i_1"'] & X_0
\end{tikzcd}$$

- In Sets, Top, Gr, Ab

$$\lim_n X_n = \{(\ldots, x_2, x_1, x_0) \mid i_n(x_n) = x_{n-1}\} \subseteq \prod_n X_n.$$

Example 1.2 (Colimit: direct limit)**.** Consider a diagram

$$X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \ldots$$

The direct limit $\operatorname{colim}_n X = \varinjlim_n X_n$ is an object in $\mathcal{C}$ with compatible maps $X_i \to \varinjlim_n X_n$ for each i and

$$\begin{tikzcd}
X_0 \arrow[r, "i_1"] \arrow[drrrrrrr] & X_1 \arrow[r, "i_2"] \arrow[drrrrrr] & \dots \arrow[r] & X_n \arrow[r, "i_{n+1}"] \arrow[drrrr] & X_{n+1} \arrow[r] \arrow[drrr] & \dots \arrow[r] & \varinjlim_n X_n \arrow[dr, dotted] & \\
& & & & & & & Z
\end{tikzcd}$$

- In Sets, Top,

$$\varinjlim_n X_n = \left(\coprod_n X_n\right) / (x \sim i_{n+1}(x))$$

Exercise 1.3. Describe the colimit of of the following direct systems in the category of abelian groups.

- $$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \dots \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \dots$$
- $$\mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^3 \to \dots \to \mathbb{Z}/p^n \xrightarrow{p} \mathbb{Z}/p^{n+1} \to \dots$$

Definition 1.4.
- Let $\mathcal{I}$ be a small category. An *$\mathcal{I}$-shaped diagram* in a category $\mathcal{C}$ is a functor $D : \mathcal{I} \to \mathcal{C}$. These form a category with morphisms natural transformations of functors. The category of $\mathcal{I}$–shaped diagrams is sometimes denoted $\mathcal{C}^{\mathcal{I}}$ or $\mathcal{I}[\mathcal{C}]$.
- Given an objection X of $\mathcal{C}$, we can always from the constant diagram $\underline{X} : \mathcal{I} \to \mathcal{C}$ which sends all objects to X and morphisms to id_X.
- A map $X \to D$ from an object X to a diagram D is a natural transformation $\underline{X} \to D$. This is called a *cone*.
- A map $D \to X$ from an diagram $D \in \mathcal{C}^{\mathcal{I}}$ to an object $X \in \mathcal{C}$ is a natural transformation $D \to \underline{X}$. This is called a *co-cone*.
- The *limit* of D is an object $\lim_{\mathcal{I}} D \in \mathcal{C}$ and a map $\lim_{\mathcal{I}} D \to D$ such that, given any $X \in \mathcal{C}$ and map $X \to D$, there is a unique map $X \to \lim_{\mathcal{I}} D$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[d, dotted] \arrow[dr] & \\
\lim_{\mathcal{I}} D \arrow[r] & D
\end{tikzcd}$$

In other words, $\lim_{\mathcal{I}} D$ is a terminal object in the category of cones.

- The *colimit* of D is an object $\operatorname{colim}_{\mathcal{I}} D \in \mathcal{C}$ and a map $D \to \operatorname{colim}_{\mathcal{I}} D$ such that, given any $X \in \mathcal{C}$ and map $D \to X$, there is a unique map $\operatorname{colim}_{\mathcal{I}} D \to X$ making the following diagram commute:

$$\begin{tikzcd} D \arrow[r] \arrow[dr] & \operatorname{colim}_{\mathcal{I}} D \arrow[d, dotted] \\ & X \end{tikzcd}$$

In other words, $\operatorname{colim}_{\mathcal{I}} D$ is an initial object in the category of co-cones.

Example 1.5.
- The product is the limit of the diagram category

$$* \qquad *$$

- The coproduct is the colimit of the diagram category

$$* \qquad *$$

- The pull-back is the limit of the diagram category

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

- The push-out is the colimit of the diagram category

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

- The inverse limit is the limit of the diagram category

$$\ldots \longrightarrow * \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow *$$

- The direct limit is the colimit of the diagram category

$$* \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow * \longrightarrow \ldots$$

Exercise 1.6. Let $\mathcal{C} = \mathrm{Ab}$. Describe the limit and the colimit of a diagram with shape:

$$* \rightrightarrows *$$

(These are called equalizers and coequalizers respectively.)

2. Some constructions as push-outs and pull-backs

Construction. *(1) Let $CX = (X \times I)/(X \times \{0\})$. This can also be described as follows. Let $i_0 : X \to X \times I$ be the map $i_0(x) = (x, 0)$. The pushout of*

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d] & X \times I \arrow[d] \\ * \arrow[r] & CX \end{tikzcd}$$

is called the cone *on X and denoted CX. Let $i : X \to CX$ be the map $i_1 : X \to X \times I$, $i_1(x) = (x, 1)$ followed by the quotient $X \times I \to CX$.*

(2) Let $SX = CX/i(X)$. This can also be described by the following pushout diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & CX \arrow[d] \\ * \arrow[r] & SX \end{tikzcd}$$

Exercise 2.1. Describe the pushout of

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

Conclude that the pushout construction is not homotopy invariant.