

# MATH 6280 - CLASS 4

## CONTENTS

1.	Limits and Colimits continued	1
2.	Some constructions as push-outs and pull-backs	4

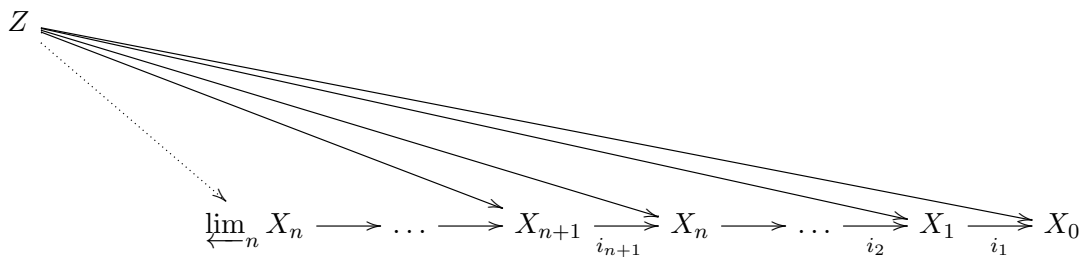
**Example 0.1.** Consider the natural numbers as a poset as follows:  $a \preceq b$  if  $a|b$ . We can turn this into a category by letting the objects be natural numbers and the  $\text{Hom}(a, b)$  be nonempty if  $a \preceq b$  and empty otherwise. Then the  $a \times b = \text{gcd}(a, b)$  and  $a \sqcup b = \text{lcm}(a, b)$ .

### 1. LIMITS AND COLIMITS CONTINUED

**Example 1.1** (Limit: inverse limit). Consider a diagram

$$\dots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow \dots \longrightarrow X_1 \longrightarrow X_0$$

The inverse limit  $\lim_n = \varprojlim_n X_n$  is an object in  $\mathcal{C}$  with compatible maps  $\varprojlim_n X_n \rightarrow X_i$  for each  $i$  and



- In Sets, Top, Gr, Ab

$$\lim_n X_n = \{(\dots, x_2, x_1, x_0) \mid i_n(x_n) = x_{n-1}\} \subseteq \prod_n X_n.$$

**Example 1.2** (Colimit: direct limit). Consider a diagram

$$X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \dots$$

The direct limit  $\text{colim}_n X = \varinjlim_n X_n$  is an object in  $\mathcal{C}$  with compatible maps  $X_i \rightarrow \varinjlim_n X_n$  for each  $i$  and

$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{i_1} & X_1 & \xrightarrow{i_2} & \dots & \longrightarrow & X_n & \xrightarrow{i_{n+1}} & X_{n+1} & \longrightarrow & \dots & \longrightarrow & \varinjlim_n X_n \\
 & & & & & & & & & & & & \searrow \text{dotted} \\
 & & & & & & & & & & & & \searrow \\
 & & & & & & & & & & & & Z
 \end{array}$$

- In Sets, Top,

$$\varinjlim_n X_n = \left( \prod_n X_n \right) / (x \sim i_{n+1}(x))$$

**Exercise 1.3.** Describe the colimit of the following direct systems in the category of abelian groups.

- 

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \dots \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \dots$$

- 

$$\mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^3 \rightarrow \dots \rightarrow \mathbb{Z}/p^n \xrightarrow{p} \mathbb{Z}/p^{n+1} \rightarrow \dots$$

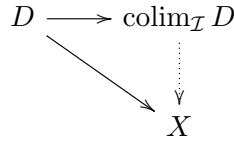
**Definition 1.4.** • Let  $\mathcal{I}$  be a small category. An  $\mathcal{I}$ -shaped diagram in a category  $\mathcal{C}$  is a functor  $D : \mathcal{I} \rightarrow \mathcal{C}$ . These form a category with morphisms natural transformations of functors. The category of  $\mathcal{I}$ -shaped diagrams is sometimes denoted  $\mathcal{C}^{\mathcal{I}}$  or  $\mathcal{I}[\mathcal{C}]$ .

- Given an object  $X$  of  $\mathcal{C}$ , we can always form the constant diagram  $\underline{X} : \mathcal{I} \rightarrow \mathcal{C}$  which sends all objects to  $X$  and morphisms to  $\text{id}_X$ .
- A map  $X \rightarrow D$  from an object  $X$  to a diagram  $D$  is a natural transformation  $\underline{X} \rightarrow D$ . This is called a *cone*.
- A map  $D \rightarrow X$  from a diagram  $D \in \mathcal{C}^{\mathcal{I}}$  to an object  $X \in \mathcal{C}$  is a natural transformation  $D \rightarrow \underline{X}$ . This is called a *co-cone*.
- The *limit* of  $D$  is an object  $\lim_{\mathcal{I}} D \in \mathcal{C}$  and a map  $\lim_{\mathcal{I}} D \rightarrow D$  such that, given any  $X \in \mathcal{C}$  and map  $X \rightarrow D$ , there is a unique map  $X \rightarrow \lim_{\mathcal{I}} D$  making the following diagram commute:

$$\begin{array}{ccc}
 X & & \\
 \vdots \searrow & \searrow & \\
 \lim_{\mathcal{I}} D & \longrightarrow & D
 \end{array}$$

In other words,  $\lim_{\mathcal{I}} D$  is a terminal object in the category of cones.

- The *colimit* of  $D$  is an object  $\text{colim}_{\mathcal{I}} D \in \mathcal{C}$  and a map  $D \rightarrow \text{colim}_{\mathcal{I}} D$  such that, given any  $X \in \mathcal{C}$  and map  $D \rightarrow X$ , there is a unique map  $\text{colim}_{\mathcal{I}} D \rightarrow X$  making the following diagram commute:



In other words,  $\text{colim}_{\mathcal{I}} D$  is an initial object in the category of co-cones.

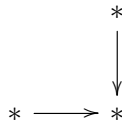
**Example 1.5.**      • The product is the limit of the diagram category



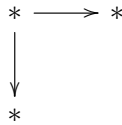
- The coproduct is the colimit of the diagram category



- The pull-back is the limit of the diagram category



- The push-out is the colimit of the diagram category



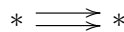
- The inverse limit is the limit of the diagram category



- The direct limit is the colimit of the diagram category



**Exercise 1.6.** Let  $\mathcal{C} = \text{Ab}$ . Describe the limit and the colimit of a diagram with shape:



(These are called equalizers and coequalizers respectively.)

## 2. SOME CONSTRUCTIONS AS PUSH-OUTS AND PULL-BACKS

**Construction.** (1) Let  $CX = (X \times I)/(X \times \{0\})$ . This can also be described as follows. Let  $i_0 : X \rightarrow X \times I$  be the map  $i_0(x) = (x, 0)$ . The pushout of

$$\begin{array}{ccc} X & \xrightarrow{i_0} & X \times I \\ \downarrow & & \downarrow \\ * & \longrightarrow & CX \end{array}$$

is called the cone on  $X$  and denoted  $CX$ . Let  $i : X \rightarrow CX$  be the map  $i_1 : X \rightarrow X \times I$ ,  $i_1(x) = (x, 1)$  followed by the quotient  $X \times I \rightarrow CX$ .

(2) Let  $SX = CX/i(X)$ . This can also be described by the following pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & CX \\ \downarrow & & \downarrow \\ * & \longrightarrow & SX \end{array}$$

**Exercise 2.1.** Describe the pushout of

$$\begin{array}{ccc} S^1 & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$

Conclude that the pushout construction is not homotopy invariant.