MATH 6280 - CLASS 38

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Dold-Thom Theorem

Let's have fun a prove a cool theorem today.

Definition 1.1. Let X be a based space with base point $*$. The n'th symmetric product $SP_n(X)$ is define as the quotient of X^n by the action of the symmetric group:

$$\mathrm{SP}^n(X) = X^n/\Sigma_n.$$

That is, $(x_1, \ldots, x_n) \sim (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every $\sigma \in \Sigma_n$. There are inclusions

$$\mathrm{SP}^n(X) \to \mathrm{SP}^{n+1}(X), \quad (x_1, \ldots, x_n) \mapsto (*, x_1, \ldots, x_n)$$

and

$$\mathrm{SP}(X) = \bigcup \mathrm{SP}^n(X).$$

This is a based space with base point the equivalence class $[*]$. Finally, it comes with a natural map $X \to \mathrm{SP}(X)$.

Remark 1.2. $\mathrm{SP}(X)$ has the union topology, but you have to be careful about the topology you put on the factors $\mathrm{SP}^n(X)$ (One must use the compactly-generated topology). One can also give a CW-structure to $\mathrm{SP}(X)$.

Claim 1.3. $\mathrm{SP}(S^2) \cong \mathbb{C}P^\infty$

Proof. Let $S^2 = \mathbb{C} \cup \{\infty\}$. Points of $\mathrm{SP}^n(S^2)$ are in bijection with unordered tuples $a_1, \ldots, a_n$ where $a_i \in \mathbb{C} \cup \{\infty\}$. The non-infty terms can be viewed as the roots of a non-zero complex polynomial of degree less than or equal to n (omit the factors $(z - \infty)$ and let $(\infty, \ldots, \infty)$ represents 1). The coefficients of the polynomial give a point $[\alpha_0 : \ldots : \alpha_n] \in \mathbb{C}P^n$ and $\mathrm{SP}^n(S^2) \cong \mathbb{C}P^n$. □

Exercise 1.4. $\mathrm{SP}(-) : \mathrm{Top}_* \to \mathrm{Top}_*$ is a functor. Further, given a homotopy $h : X \times I \to Y$ between f, g, we get a homotopy $h : \mathrm{SP}(X) \times I \to \mathrm{SP}(Y)$ so that $\mathrm{SP}(f) \simeq \mathrm{SP}(g)$. Therefore, if $X \simeq Y$, then $\mathrm{SP}(X) \simeq \mathrm{SP}(Y)$ and if $X \simeq *$, then $\mathrm{SP}(X) \simeq \mathrm{SP}(*) \simeq *$.

Claim 1.5. *$S^1 \to \mathrm{SP}(S^1)$ is a homotopy equivalence.*

Proof. Let $S^1 = \mathbb{C} - \{0, \infty\}$. Then as before, $\mathrm{SP}^n(S^1)$ are those polynomials that have roots $a_1, \ldots, a_n$ with $a_i \neq 0, \infty$. So they are polynomials of degree exactly n with non-zero constant term. So we can think of these as the points $[\alpha_0 : \ldots : \alpha_n]$ in $\mathbb{C}P^n$ such that $\alpha_0, \alpha_n \neq 0$. Scaling so that $\alpha_0 = 1$, this is equivalent to tuples in $\mathbb{C}^{n-1} \times (\mathbb{C} \backslash \{0\}) \simeq S^1$. So each $S^1 \to \mathrm{SP}^n(S^1)$ are homotopy equivalences, hence $S^1 \to \mathrm{SP}(S^1)$ is a weak homotopy equivalence. Since both are CW-complexes, is is a homotopy equivalence. □

Theorem 1.6 (Dold-Thom). *For X path connected, $\pi_n \mathrm{SP}(X) \cong \widetilde{H}_n(X; \mathbb{Z})$. For*

$$\pi_{*+1} \mathrm{SP}(\Sigma(-)) : \mathrm{CWTop}_* \to \mathrm{Ab}$$

is a reduced homology theory which satisfies the dimension axiom.

Remark 1.7. It follows from the Dold-Thom theorem that $\mathrm{SP}(S^n) \cong K(\mathbb{Z}, n)$. Similarly, $\mathrm{SP}(M(G, n)) \cong K(G, n)$. This turns out to be a definition for $K(\mathbb{Z}, n)$ which is fairly convenient to generalize.

(1) Dimension
(2) Exactness
(3) Suspension
(4) Additivity

Definition 1.8. A map $p : E \to B$ is a *quasi-fibration* if for every $b \in B$ and $e \in F_p = p^{-1}(b)$,

$$p_* : \pi_*(E, F_b, e) \xrightarrow{\cong} \pi_*(B).$$

Note that it is exactly for quasi-fibrations that we get a long exact sequence

$$\ldots \to \pi_*(F_b, e) \to \pi_*(E, e) \to \pi_*(B, b) \to \pi_{*-1}(F_b, e) \to \ldots.$$

The key to proving the theorem is the following proposition:

Proposition 1.9. *Let (A, X) be a based CW-pair with A path-connected and $p : X \to X/A$. Then*

$$\mathrm{SP}(p) : \mathrm{SP}(X) \to \mathrm{SP}(X/A)$$

is a quasi-fibration whose fiber at every point is homotopy equivalent to $\mathrm{SP}(A)$.

If X is path connected, from $X \to CX \to \Sigma X$ and the induced long exact sequence on $\mathrm{SP}(X) \to \mathrm{SP}(CX) \to \mathrm{SP}(\Sigma X)$, we get a natural isomorphism

$$\pi_{n+1}(\mathrm{SP}(\Sigma X)) \xrightarrow{\cong} \pi_n(\mathrm{SP}(X))$$

giving us the suspension axiom, and similarly, we also get exactness.

The dimension axiom comes from noting that $\mathrm{SP}^n(S^0)$ is a discrete set with $n+1$ elements. So $\pi_0(\mathrm{SP}(S^0))$ is a countable set which obtains the structure of an abelian group the isomorphisms

$$\pi_1(\mathrm{SP}(S^1)) \cong \pi_2(\mathrm{SP}(S^2)) \cong \pi_2(\mathbb{C}P^\infty) \cong \mathbb{Z}.$$

For additivity, one needs the following proposition.

Proposition 1.10. *Let $X = \operatorname{colim} X_\lambda$ be a filtered colimit where the X_λ are closed, respectively open subspaces of X containing the base point, then so is $\mathrm{SP}(X_\lambda) \to \mathrm{SP}(X)$ and $\operatorname{colim} \mathrm{SP}(X_\lambda) \to \mathrm{SP}(X)$ is a weak equivalence.*

Proof Sketch. Since $X_\lambda^n \to X^n$ is closed, resp. open, this passes to the quotients $\mathrm{SP}^n(X_\lambda) \to \mathrm{SP}^n(X)$. Since $\mathrm{SP}(X)$ has the union topology and $\mathrm{SP}(X_\lambda) \cap \mathrm{SP}^n(X) = \mathrm{SP}^n(X_\lambda)$, it follows that $\mathrm{SP}(X_\lambda) \to \mathrm{SP}(X)$ are closed, resp. open, inclusions. It's obvious that $\operatorname{colim} \mathrm{SP}(X_\lambda) \to \mathrm{SP}(X)$ is a continuous bijection, but to finish the proof, one has to look at the topology on $\mathrm{SP}(X)$ and conclude that the inverse is continuous on compact sets. $\square$

Corollary 1.11. *If $X = \bigvee_{\lambda \in \Lambda} X_\lambda$, then $\pi_* \mathrm{SP}(X) \cong \bigoplus_\lambda \pi_* \mathrm{SP}(X_\lambda)$.*

Proof. First, we get it for $X_1 \vee X_2$. There's a long exact sequence

$$\ldots \to \pi_* \mathrm{SP}(X_1) \to \pi_* \mathrm{SP}(X_1 \vee X_2) \to \pi_* \mathrm{SP}(X_2) \to \ldots$$

and the map $\pi_* \mathrm{SP}(X_1 \vee X_2) \to \pi_* \mathrm{SP}(X_2)$ has a splitting. So, the long exact splits as

$$\pi_* \mathrm{SP}(X_1 \vee X_2) \cong \pi_*(\mathrm{SP}(X_1)) \oplus \pi_*(\mathrm{SP}(X_2)).$$

By induction, this gives it for finite wedges.

Now, we consider the filtered system $\Gamma \subset \Lambda$, Γ finite. Then $X = \operatorname{colim}_{\Gamma \subset \Lambda} \bigvee_{\lambda \in \Gamma} X_\lambda$. Then

$$\pi_* \mathrm{SP}(X) \cong \operatorname{colim} \pi_*(\mathrm{SP}(X_\lambda)) \cong \operatorname{colim}_{\Gamma \subset \Lambda} \bigoplus_{\lambda \in \Gamma} \pi_* \mathrm{SP}(X_\lambda) \cong \bigoplus_{\lambda \in \Lambda} \pi_* \mathrm{SP}(X_\lambda)$$

$\square$

Remark 1.12. It follows from this that

$$\mathrm{SP}(X \vee Y) \to \mathrm{SP}(X) \times \mathrm{SP}(Y)$$

is a weak homotopy equivalence.

Remark 1.13. With the right topologies, $\mathrm{SP}(X)$ is a topological monoid where

$$\mathrm{SP}(X) \times \mathrm{SP}(X) \to \mathrm{SP}(X)$$

is given as maps $\mathrm{SP}^n(X) \times \mathrm{SP}^m(X) \to \mathrm{SP}^{n+m}(X)$ by

$$((x_1, \dots, x_n), (y_1, \dots, y_m)) \mapsto (x_1, \dots, x_n, y_1, \dots, y_m).$$