MATH 6280 - CLASS 36

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Remarks on Künneth continued

Recall from last time:

Corollary 1.1 (Künneth Isomorphism)**.** *If k is a field, then for any space X and Y, there is a natural isomorphism*

$$\bigoplus_{p+q=n} H_p(X;k)\otimes H_q(Y;k) \xrightarrow{\cong} H_n(X\times Y;k),$$

i.e., $H_*(X;k)\otimes H_*(Y;k)\cong H_*(X\times Y;k)$.

Proof. $H_*(Z;k)$ is free as a k–module for any space Z. Hence, all the Tor_1^k terms vanish. The map on chains inducing the isomorphism is natural in both X and Y, hence this so is the induced map on cohomology. $\square$

Exercise 1.2. Let X and Y be based spaces. Prove that there is a map

$$\widetilde{H}_*(X;R)\otimes \widetilde{H}_*(Y;R)\to \widetilde{H}_n(X\wedge Y;R)$$

which is an isomorphism when is a field.

Culture Moment. A cohomology theory with the property that $\widetilde{E}_*(X)$ is free over $\widetilde{E}_*(S^0)$ for any X is called a *field*. One can show that such theories satisfy the Künneth isomorphism. However, fields are rather rare. Of course, $\widetilde{H}^*(-;k)$ for any field k are fields. Another example is mod-p K–theory. For each prime p, there are certain reduced homology theories $K(n)_*(-)$ for each integer n, called the Morava K–theories, such that all other fields are direct sums and shifts of the $\widetilde{H}^*(-;k)$ and the $K(n)_*(-)$.

Now, note that there is a map of chain complexes

$$\mathrm{Hom}(C_*(X),R)\otimes\mathrm{Hom}(C_*(Y),R)\to\mathrm{Hom}(C_*(X)\otimes C_*(Y),R)\cong C^*(X\times Y;R)$$

where

$$(\phi\otimes\phi')(x\otimes x')=\phi(x)\phi'(x').$$

As long as X has finitely many cells in each degree, this is an isomorphism. Then, we can apply the Künneth theorem:

Theorem 1.3. *Let X be a CW-complex with finitely many cells in each degree and R be a PID. Then there is an exact sequence*

$$0\to\bigoplus_{p+q=n}H^p(X;R)\otimes H^q(Y;R)\to H^n(X\times Y;R)\to\bigoplus_{p+q=n+1}\mathrm{Tor}_1^R(H^p(X;R),H^q(Y;R))\to 0.$$

2. Universal Coefficients

Now, let's see what the Universal Coefficient Theorem for cohomology gives us. Again, $C_*(X;R)$ is always a chain complex of free R–modules, hence applying the result to this chain complex, we get

$$0\to\mathrm{Ext}^1_R(H_{n-1}(X),M)\to H^n(X;M)\xrightarrow{\alpha}\mathrm{Hom}_R(H_n(X),M)\to 0.$$

which relates homology and cohomology. In particular, if we take $R=M=\mathbb{Z}$, then

$$0\to\mathrm{Ext}(H_{n-1}(X),\mathbb{Z})\to H^n(X)\xrightarrow{\alpha}\mathrm{Hom}(H_n(X),\mathbb{Z})\to 0.$$

If X has finitely many cells in each degree, then $H_{n-1}(X)$ is a finitely generated abelian group. Hence, for each n,

$$H_n(X)\cong\mathrm{free}(H_n(X))\oplus\mathrm{torsion}(H_n(X)),$$

Further, (exercise!)

$$\mathrm{Ext}(H_{n-1}(X),\mathbb{Z})\cong\mathrm{torsion}(H_{n-1}(X))\quad\text{and}\quad\mathrm{Hom}(H_n(X),\mathbb{Z})\cong\mathrm{free}(H_n(X)).$$

Using the splitting, we get a non-natural isomorphism

$$H^n(X) \cong \mathrm{torsion}(H_{n-1}(X)) \oplus \mathrm{free}(H_n(X)).$$

So, in this case, the homology determines the cohomology.

Next, consider the case when $R = k$ for a field k.

Corollary 2.1. *Let k be a field, then there is a natural isomorphism*

$$H^n(X; k) \xrightarrow{\cong} (H_n(X; k))^*$$

where for a vector space V, $V^* = \mathrm{Hom}_k(V, k)$ denotes the dual.

Proof. $H_{n-1}(X; k)$ is free as a k–module hence the Ext–term vanishes. □

3. Cup products

Suppose that X and X' are CW complexes and A and A' are abelian groups. Then

$$C^*(X \times X'; A \otimes A') \cong \mathrm{Hom}(C_*(X) \otimes C_*(X'), A \otimes A').$$

Further, there is a map of chain complexes

$$\mathrm{Hom}(C_*(X), A) \otimes \mathrm{Hom}(C_*(X'), A') \to \mathrm{Hom}(C_*(X) \otimes C_*(X'), A \otimes A') \cong C^*(X \times X'; A \otimes A')$$

where

$$(\phi \otimes \phi')(x \otimes x') = \phi(x) \otimes \phi'(x').$$

Further, if we have a map of abelian groups $A \otimes A' \to B$ and a map $Y \to X \times X'$, we get maps

$$\mathrm{Hom}(C_*(X), A) \otimes \mathrm{Hom}(C_*(X'), A') \to C^*(X \times X'; A \otimes A') \to C^*(X \times X'; B) \to C^*(Y; B).$$

This induces a maps on cohomology:

$$H^*(X; A) \otimes H^*(X'; A') \to H^*(X \times X'; A \otimes A') \to H^*(X \times X'; B) \to H^*(Y; B).$$

Now, if $X = X'$ and $R = A = A' = B$ a ring with the map $R \otimes R \to R$ given by multiplication and a map $\Delta : X \to X \times X$ given by the diagonal, so we get a map

$$\smile : H^*(X; R) \otimes H^*(X; R) \to H^*(X; R)$$

called the *cup product* and written as

$$xy = x \smile y$$

where for $x \in H^p$ and $y \in H^q$, we have that $x \smile y \in H^{p+q}$.

Let $X \to *$ be the unique map. Then this gives a map $R \cong H^*(*; R) \to H^*(X; R)$. We let $1 \in H^*(X; R)$ be the image of $1 \in R$. Then the cup product is unital, associative and graded commutative:

$$xy = (-1)^{\deg x \deg y} yx,$$

making $H^*(X; R)$ into a graded commutative ring. The cup product is also natural. If

$$f : X \to Y$$

then $f(xy) = f(x)f(y)$.

Remark 3.1. Note that Δ is not a cellular map, so to compute the cup product, one must first approximate it by a cellular map and this can be a pain.

Remark 3.2. The map

$$H^*(X; A) \otimes H^*(X'; A') \to H^*(X \times X'; A \otimes A')$$

is called the *external product*.

Exercise 3.3. When the Künneth map $H^*(X; R) \otimes H^*(Y; R) \to H^*(X \times Y; R)$ is a map of rings.

Example 3.4.
- $H^*(S^n; \mathbb{Z}) = \mathbb{Z}[x]/x^2$ for $|x| = n$
- $H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[x]$, $|x| = 2$
- $H^*(\mathbb{R}P^\infty; \mathbb{Z}) = \mathbb{Z}[x]/(2x)$, $|x| = 2$
- $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x]$, $|x| = 2$

4. Reduced cup products

Now, consider reduced cohomology $\widetilde{H}(X; R)$.

Claim 4.1. $\widetilde{C}(X; R) \otimes_R \widetilde{C}(Y; R) \cong \widetilde{C}(X \wedge Y; R)$.

Further, there is a diagonal

$$\widetilde{\Delta} : X \xrightarrow{\Delta} X \times X \to X \wedge X.$$

Therefore, we get a reduced cup product:

$$\smile : \widetilde{H}(X; R) \otimes \widetilde{H}(X; R) \to \widetilde{H}(X; R).$$