# MATH 6280 - CLASS 35

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These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

## 1. Remarks on Künneth continued

Recall from last time:

**Corollary 1.1** (Künneth Isomorphism)**.** *If $k$ is a field, then for any space $X$ and $Y$, there is a natural isomorphism*

$$\bigoplus_{p+q=n} H_p(X;k) \otimes H_q(Y;k) \xrightarrow{\cong} H_n(X \times Y;k),$$

*i.e.,* $H_*(X;k) \otimes H_*(Y;k) \cong H_*(X \times Y;k)$.

*Proof.* $H_*(Z;k)$ is free as a $k$–module for any space $Z$. Hence, all the $\mathrm{Tor}_1^k$ terms vanish. The map on chains inducing the isomorphism is natural in both $X$ and $Y$, hence this so is the induced map on cohomology. □

**Exercise 1.2.** Let $X$ and $Y$ be based spaces. Prove that there is a map

$$\widetilde{H}_*(X;R) \otimes \widetilde{H}_*(Y;R) \to \widetilde{H}_n(X \wedge Y;R)$$

which is an isomorphism when is a field.

**Culture Moment.** A cohomology theory with the property that $\widetilde{E}_*(X)$ is free over $\widetilde{E}_*(S^0)$ for any $X$ is called a *field*. One can show that such theories satisfy the Künneth isomorphism. However, fields are rather rare. Of course, $\widetilde{H}^*(-;k)$ for any field $k$ are fields. Another example is mod-$p$ $K$–theory. For each prime $p$, there are certain reduced homology theories $K(n)_*(-)$ for each integer $n$, called the Morava $K$–theories, such that all other fields are direct sums and shifts of the $\widetilde{H}^*(-;k)$ and the $K(n)_*(-)$.

Now, note that there is a map of chain complexes

$$\mathrm{Hom}(C_*(X),R)\otimes \mathrm{Hom}(C_*(Y),R)\to \mathrm{Hom}(C_*(X)\otimes C_*(Y),R)\cong C^*(X\times Y;R)$$

where

$$(\phi\otimes\phi')(x\otimes x')=\phi(x)\phi'(x').$$

As long as $X$ has finitely many cells in each degree, this is an isomorphism. Then, we can apply the Künneth theorem:

**Theorem 1.3.** *Let $X$ be a CW-complex with finitely many cells in each degree and $R$ be a PID. Then there is an exact sequence*

$$0\to \bigoplus_{p+q=n} H^p(X;R)\otimes H^q(Y;R)\to H^n(X\times Y;R)\to \bigoplus_{p+q=n+1}\mathrm{Tor}_1^R(H^p(X;R),H^q(Y;R))\to 0.$$

## 2. Universal Coefficients

Now, let's see what the Universal Coefficient Theorem for cohomology gives us. Again, $C_*(X;R)$ is always a chain complex of free $R$–modules, hence applying the result to this chain complex, we get

$$0\to \mathrm{Ext}^1_R(H_{n-1}(X),M)\to H^n(X;M)\xrightarrow{\alpha}\mathrm{Hom}_R(H_n(X),M)\to 0.$$

which relates homology and cohomology. In particular, if we take $R=M=\mathbb{Z}$, then

$$0\to \mathrm{Ext}(H_{n-1}(X),\mathbb{Z})\to H^n(X)\xrightarrow{\alpha}\mathrm{Hom}(H_n(X),\mathbb{Z})\to 0.$$

If $X$ has finitely many cells in each degree, then $H_{n-1}(X)$ is a finitely generated abelian group. Hence, for each $n$,

$$H_n(X)\cong \mathrm{free}(H_n(X))\oplus \mathrm{torsion}(H_n(X)),$$

Further, (exercise!)

$$\mathrm{Ext}(H_{n-1}(X),\mathbb{Z})\cong \mathrm{torsion}(H_{n-1}(X))\quad\text{and}\quad \mathrm{Hom}(H_n(X),\mathbb{Z})\cong \mathrm{free}(H_n(X)).$$

Using the splitting, we get a non-natural isomorphism

$$H^n(X) \cong \text{torsion}(H_{n-1}(X)) \oplus \text{free}(H_n(X)).$$

So, in this case, the homology determines the cohomology.

Next, consider the case when $R = k$ for a field $k$.

**Corollary 2.1.** *Let $k$ be a field, then there is a natural isomorphism*

$$H^n(X;k) \xrightarrow{\cong} (H_n(X;k))^*$$

*where for a vector space $V$, $V^* = \text{Hom}_k(V,k)$ denotes the dual.*

*Proof.* $H_{n-1}(X;k)$ is free as a $k$–module hence the Ext–term vanishes. □

## 3. Cup products

Suppose that $X$ and $X'$ are CW complexes and $A$ and $A'$ are abelian groups. Then

$$C^*(X \times X'; A \otimes A') \cong \text{Hom}(C_*(X) \otimes C_*(X'), A \otimes A').$$

Further, there is a map of chain complexes

$$\text{Hom}(C_*(X), A) \otimes \text{Hom}(C_*(X'), A') \to \text{Hom}(C_*(X) \otimes C_*(X'), A \otimes A') \cong C^*(X \times X'; A \otimes A')$$

where

$$(\phi \otimes \phi')(x \otimes x') = \phi(x) \otimes \phi'(x').$$

Further, if we have a map of abelian groups $A \otimes A' \to B$ and a map $Y \to X \times X'$, we get maps

$$\text{Hom}(C_*(X), A) \otimes \text{Hom}(C_*(X'), A') \to C^*(X \times X'; A \otimes A') \to C^*(X \times X'; B) \to C^*(Y; B).$$

This induces a maps on cohomology:

$$H^*(X;A) \otimes H^*(X';A') \to H^*(X \times X'; A \otimes A') \to H^*(X \times X'; B) \to H^*(Y;B).$$

Now, if $X = X'$ and $R = A = A' = B$ a ring with the map $R \otimes R \to R$ given by multiplication and a map $\Delta : X \to X \times X$ given by the diagonal, so we get a map

$$\smile : H^*(X;R) \otimes H^*(X;R) \to H^*(X;R)$$

called the *cup product* and written as

$$xy = x \smile y$$

where for $x \in H^p$ and $y \in H^q$, we have that $x \smile y \in H^{p+q}$.

Let $X \to *$ be the unique map. Then this gives a map $R \cong H^*(*;R) \to H^*(X;R)$. We let $1 \in H^*(X;R)$ be the image of $1 \in R$. Then the cup product is unital, associative and graded commutative:

$$xy = (-1)^{\deg x \deg y} yx,$$

making $H^*(X;R)$ into a graded commutative ring. The cup product is also natural. If

$$f : X \to Y$$

then $f(xy) = f(x)f(y)$.

**Remark 3.1.** Note that $\Delta$ is not a cellular map, so to compute the cup product, one must first approximate it by a cellular map and this can be a pain.

**Remark 3.2.** The map

$$H^*(X;A) \otimes H^*(X';A') \to H^*(X \times X'; A \otimes A')$$

is called the *external product*.

**Exercise 3.3.** When the Künneth map $H^*(X;R) \otimes H^*(Y;R) \to H^*(X \times Y;R)$ is a map of rings.

**Example 3.4.**
- $H^*(S^n;\mathbb{Z}) = \mathbb{Z}[x]/x^2$ for $|x| = n$
- $H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[x]$, $|x| = 2$
- $H^*(\mathbb{R}P^\infty;\mathbb{Z}) = \mathbb{Z}[x]/(2x)$, $|x| = 2$
- $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[x]$, $|x| = 2$

## 4. Reduced cup products

Now, consider reduced cohomology $\widetilde{H}(X;R)$.

**Claim 4.1.** $\widetilde{C}(X;R) \otimes_R \widetilde{C}(Y;R) \cong \widetilde{C}(X \wedge Y;R)$.

Further, there is a diagonal

$$\widetilde{\Delta} : X \xrightarrow{\Delta} X \times X \to X \wedge X.$$

Therefore, we get a reduced cup product:

$$\smile : \widetilde{H}(X;R) \otimes \widetilde{H}(X;R) \to \widetilde{H}(X;R).$$