MATH 6280 - CLASS 34

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Colimits Continued

As for homology, there is a Mayer-Vietoris theorem for generalized cohomologies:

Theorem 1.1. *(Mayer-Vietoris for Cohomology) Let $E^* : \mathrm{CWTop\,pairs}^{op} \to \mathrm{Ab}$. Let $(X; A, B)$ be a CW-triad and i denote the various inclusions. Let $C = A \cap B$.*

Then there is a long exact sequence

$$\ldots \to E^{*-1}(C) \xrightarrow{\delta} E^*(X) \xrightarrow{(i^*, i^*)} E^*(A) \oplus E^*(B) \xrightarrow{i_* \oplus - i_*} E^*(C) \to \ldots$$

where δ is the composit:

$$E^{*-1}(C) \xrightarrow{\partial} E^*(A, C) \xrightarrow[exc.]{\cong} E^*(X, B) \xrightarrow{i^*} E^*(X)$$

Let $X = \bigcup_{i=i}^{\infty} X_i$ for $j_i : X_i \subseteq X_{i+1}$. Recall that for $E_* : \mathrm{Top}\mathit{pairs} \to \mathit{Ab}$ a generalized homology theory, then the natural map

$$\operatorname{colim} E_*(X_i) \to E_*(X)$$

is an isomorphism.

Now, let $E^* : \mathrm{Toppairs}^{op} \to \mathit{Ab}$ be a generalized cohomology theory. The natural map now goes in the other direction and the colimit is replaced by an inverse limit:

$$E^*(X) \to \lim E^*(X_i).$$

This map is not in general an isomorphism. Rather, we have

Theorem 1.2. *There are natural exact sequences*

$$0 \to \lim{}^1 E^{q-1}(X_i) \to E^q(X) \to \lim E^q(X_i) \to 0.$$

Let's explain what this $\lim{}^1$-term means. An *equalizer*, is the limit of a diagram:

$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

Let $\mathcal{A}$ be a category with products and equilizers. For $f_{i+1} : A_{i+1} \to A_i$ a inverse system, $\lim A_i$ is the coequilizer:

$$\lim A_i \longrightarrow \prod A_i \overset{\prod id}{\underset{\prod f_i}{\rightrightarrows}} \prod A_i$$

In abelian groups, these diagrams can be rewritten as an exact sequence

$$0 \to \lim A_i \to \prod_{i=1}^{\infty} A_i \xrightarrow{s} \prod_{i=1}^{\infty} A_i \to \lim{}^1 A_i \to 0 \tag{1}$$

for $s_{i+1} : A_{i+1} \to A_i$ given by $s_{i+1}(a) = a - f_{i+1}(a)$ and $s = \prod s_i$. As (1) suggests, the map s is not necessarily surjective and $\lim{}^1$ measures that failure.

Remark 1.3. For N the poset $0 \to 1 \to 2 \to \ldots$, the functor $\lim : \mathrm{Ab}^{\mathsf{N}^{op}} \to \mathrm{Ab}$ is a left exact functor and $\lim{}^1$ is the first right derived functor. In contrast, $\mathrm{colim} : \mathrm{Ab}^{\mathsf{N}} \to \mathrm{Ab}$ is exact.

Proof. This is similar to the proof for the colimits, but the lack of exactness gives it an extra twist. As before, Let

$$\mathrm{tel} X_i = \bigcup_{i=0}^{\infty} X_i \times [i, i+1]$$

and divide it as $(\mathrm{tel} X_i; A, B)$ such that

$$A \xrightarrow{r} \bigcup_{i \geq 1} X_{2i-1} \qquad B \xrightarrow{r} \bigcup_{i \geq 0} X_{2i} \qquad C = A \cap B \xrightarrow{r} \bigcup_{i \geq 0} X_i$$

are weak equivalences. Construct a commutative diagram using Mayer-Vietoris such that the bottom row is the (1).

$$\begin{tikzcd}
\dots \arrow[r] & E^*(\mathrm{tel}X_i) \arrow[r] \arrow[d, "\cong"] & E^*(A)\oplus E^*(B) \arrow[r] \arrow[d, "\cong"] & E^*(C) \arrow[r] \arrow[d, "\cong"] & E^{*+1}(\mathrm{tel}X_i) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & E^*(X) \arrow[r] \arrow[d] & \prod_{i\geq 0} E^*(X_i) \arrow[r] \arrow[d, "\cong"] & \prod_{i\geq 0} E^*(X_i) \arrow[r] \arrow[d, "\cong"] & E^{*+1}(X) \arrow[r] & \dots \\
0 \arrow[r] & \lim E^*(X_i) \arrow[r] & \prod_{i\geq 0} E_*(X_i) \arrow[r] & \prod_{i\geq 0} E_*(X_i) \arrow[r] & \lim{}^1 E^*(X_i) \arrow[r] & 0
\end{tikzcd}$$

Recall that if

$$\dots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} A^{i+2} \xrightarrow{f^{i+2}} A^{i+3} \longrightarrow \dots$$

is exact, then there is an exact sequence

$$0 \to \mathrm{coker}(f^{i-1}) \xrightarrow{\overline{f}^i} A^{i+1} \xrightarrow{\underline{f}^{i+1}} \ker(f^{i+2}) \to 0$$

where $\overline{f}^i$ and $\underline{f}^{i+1}$ are the universal maps induced by f^i and f^{i+1} respectively.

Hence, since $\lim E^*(X_i)$ is the kernel of the middle arrow, we have a surjection

$$E^*(X) \to \lim E^*(X_i)$$

is a surjection. Further, $\lim{}^1 E^*(X_i)$ is the cokernel of the middle arrow. Therefore, by the exactness of the sequence, it is the kernel of the map

$$E^{*+1}(X) \to \lim E^{*+1}(X_i) \to 0.$$

Hence, gluing these facts together proves the claim. □

2. Consequences of the Künneth and Universal Coefficient Theorem for spaces

Theorem 2.1. *Let R be a PID. Let X be a free chain complex of R–modules.*

(a) (Künneth Theorem) Let Y be any chain complex of R–modules. Then there is a short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(X)\otimes_R H_q(Y) \to H_n(X\otimes_R Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X), H_q(Y)) \to 0.$$

This sequence is split but the splitting is not natural.

(b) Let Y be any cochain complex of R–modules. Then there is a short exact sequence

$$0 \to \prod_{p+q=n-1} \mathrm{Ext}^1_R(H_p(X), H^q(Y)) \to H^n(\mathrm{Hom}_R(X,Y)) \to \prod_{p+q=n} \mathrm{Hom}_R(H_p(X), H^q(Y)) \to 0.$$

This sequence is split but the splitting is not natural.

Corollary 2.2 (Universal Coefficient Theorems)**.** *Let R be a PID. Let X a free chain complex of R–modules and M be any R–module.*

(a) There is a not naturally split short exact sequence

$$0 \to H_n(X) \otimes_R M \xrightarrow{\alpha} H_n(X \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(X), M) \to 0.$$

The map α is defined by

$$\alpha([x] \otimes m) = [x \otimes m]$$

where $x \in X$ is a representative for $[x]$.

(b) There is a not naturally split short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X), M) \to H^n(\mathrm{Hom}_R(X, M)) \xrightarrow{\alpha} \mathrm{Hom}_R(H_n(X), M) \to 0.$$

The map α is defined by

$$\alpha([f])([x]) = [f(x)]$$

where $f \in \mathrm{Hom}(X, M)$ is a representative for $[f]$ and $x \in X$ is a representative for $[x]$.

Exercise 2.3. Go over the proof of (b), the universal coefficient theorem for cohomology.

Recall that for X and Y CW-complexes, there is an isomorphism

$$C_*(X;R) \otimes_R C_*(Y;R) \to C_*(X \times Y; R).$$

Further, $C_*(X;R)$ is always a chain complex of free R–modules. Therefore,

$$H_*(X \times Y; R) \cong H_*(C_*(X;R) \otimes_R C_*(Y;R)).$$

Hence, the Künneth can be restated as

$$0 \to \bigoplus_{p+q=n} H_p(X;R) \otimes H_q(Y;R) \to H_n(X \times Y; R) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X;R), H_q(Y;R)) \to 0.$$

Corollary 2.4 (Künneth Isomorphism)**.** *If k is a field, then for any space X and Y, there is a natural isomorphism*

$$\bigoplus_{p+q=n} H_p(X;k) \otimes H_q(Y;k) \xrightarrow{\cong} H_n(X \times Y; k),$$

i.e., $H_*(X;k) \otimes H_*(Y;k) \cong H_*(X \times Y;k)$.

Proof. $H_*(Z;k)$ is free as a k–module for any space Z. Hence, all the Tor_1^k terms vanish. The map on chains inducing the isomorphism is natural in both X and Y, hence this so is the induced map on cohomology. $\square$