MATH 6280 - CLASS 33

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Preview

Let $X = \bigcup_{i=0}^{\infty} X_0$ for $j_i : X_i \subseteq X_{i+1}$. Recall that since S^n and $S^n \times I$ are compact, we have

$$\operatorname{colim} \pi_*(X_i) \to \pi_*(X)$$

The next goal is to prove that this holds for generalized homology theories as well.

Theorem 1.1. *Let $E_* : \mathrm{Top}pairs \to Ab$ be a generalized homology theory, then the natural map*

$$\operatorname{colim} E_*(X_i) \to E_*(X)$$

is an isomorphism.

We start with some remarks.

Remark 1.2. Recall that a coequlizer is the colimit of a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

or equivalently, the pushout:

$$\begin{tikzcd} A\coprod A \arrow[r, "f\sqcup g"] \arrow[d, "\nabla"'] & B \ . \\ A & \end{tikzcd}$$

Let $\mathcal{A}$ be a category with coproducts and co-equilizers. For $f_i : A_i \to A_{i+1}$ a directed system, $\operatorname{colim} A_i$ is the coequilizer:

$$\coprod A_i \underset{\sqcup f_i}{\overset{\sqcup id}{\rightrightarrows}} \coprod A_i \longrightarrow \operatorname{colim} A_i$$

(In abelian groups, these diagrams can be rewritten as an exact sequence

$$0 \to \bigoplus_{i=1}^{\infty} A_i \xrightarrow{s} \bigoplus_{i=1}^{\infty} A_i \to \operatorname{colim} A_i \to 0$$

for $s_i : A_i \to A_{i+1}$ given by $s_i(a) = a - f_i(a)$ and $s = \oplus s_i$.)

Therefore, if a functor preserves arbitrary coproducts and coequilizers, then it preserves directed colimits. Generalized homology theories do preserve coproducts, but we need some kind of approximation of what it would been for it to preserve co-equiliizers, or at least, how it behaves with pushouts. This is what Mayer-Vietoris will gives us.

2. Mayer-Vietoris

There are many versions of this theorem. I'll give a simple one and let you figure out others.

Theorem 2.1. *Let $E_* : \mathrm{CWTop\,pairs} \to \mathrm{Ab}$. Let $(X; A, B)$ be a CW-triad and i denote the various inclusions. Let $C = A \cap B$.*

Then there is a long exact sequence

$$\ldots \to E_*(C) \xrightarrow{(i_*, i_*)} E_*(A) \oplus E_*(B) \xrightarrow{i_* \oplus - i_*} E_*(X) \xrightarrow{\delta} E_{*-1}(C) \to \ldots$$

where δ is the composit:

$$E_*(X) \longrightarrow E_*(X, B) \underset{exc.}{\xrightarrow{\cong}} E_*(A, C) \xrightarrow{\partial} E_{*-1}(C).$$

Remark 2.2. Under these conditions, we have that

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

is a (homotopy) pushout and MVT says that $E_*(-)$ turns this into a long exact sequence.

First, note the following lemma.

Lemma 2.3. *Let* $E_* : \mathrm{CWTop\,pairs} \to \mathrm{Ab}$*. Let* $(X; A, B)$ *be a CW-triad and* i *denote the various inclusions. Let* $C = A \cap B$*. Then*

$$E_*(A,C) \oplus E_*(B,C) \to E_*(X,C)$$

is an isomorphism. Further, the inclusion of the factors is split by the excision isomorphism:

$$E_*(A,C) \longrightarrow E_*(X,C) \longrightarrow E_*(X,B) \xrightarrow[exc.]{\cong} E_*(A,C) \ ,$$

and equivalently,

$$E_*(X,C) \xrightarrow{i_* \oplus i_*} E_*(X,A) \oplus E_*(X,B)$$

is an isomorphism.

Proof. We have $X/C \simeq A/C \vee B/C$. Then

$$\begin{aligned} E_*(A,C) \oplus E_*(B,C) &= \widetilde{E}_*(A/C) \oplus \widetilde{E}_*(B/C) \\ &\xrightarrow{i_* \oplus i_*}{}^{\cong} \widetilde{E}_*(A/C \vee B/C) \\ &\cong \widetilde{E}_*(X/C) \\ &= E_*(X,C). \end{aligned}$$

For CW triad, the excision isomorphism comes from the equivalence $X/B \cong A/C$ and the second claim follows. $\square$

Proof of Theorem 2.1. As in *Concise*, the following diagram commutes:

$$\begin{tikzcd}
 & E_*(A) \arrow[dr] & & E_*(X,B) & & E_*(A,C) \arrow[ll, "\cong"'] \arrow[dl] \arrow[dr, "\partial"] & \\
E_*(C) \arrow[ur] \arrow[rr] \arrow[dr] & & E_*(X) \arrow[ur, "j_*"] \arrow[rr, "k_*"] \arrow[dr, "j_*"'] & & E_*(X,C) \arrow[ul] \arrow[dl] \arrow[rr, "\partial"] & & E_{*-1}(C) \\
 & E_*(B) \arrow[ur] & & E_*(X,A) & & E_*(B,C) \arrow[ll, "\cong"] \arrow[ul] \arrow[ur, "\partial"'] &
\end{tikzcd}$$

We use the notation $a \in A$, $b \in B$, $c \in C$ and $x \in X$

We first prove exactness at

$$E_*(C) \xrightarrow{(i_*, i_*)} E_*(A) \oplus E_*(B) \xrightarrow{i_* \oplus -i_*} E_*(X).$$

That the composition is zero is obvious. Suppose $i_*(a) - i_*(b) = 0$. Then $i_*(a) = i_*(b)$. Then, by exactness of

$$E_*(A) \to E_*(X) \xrightarrow{j_*} E_*(X, A)$$

we have $j_*(i_*(a)) = 0 \in E_*(X, A)$ and similarly, $j_*(i_*(b)) = 0 \in E_*(X, B)$.

Since $E_*(X, C) \cong E_*(X, A) \oplus E_*(X, B)$

$$k_*(i_*(a)) = k_*(i_*(b)) = 0.$$

There is a $c \in E_*(C)$ such that $i_*(c) = i_*(a) = i_*(b) \in E_*(X)$. Hence, $(a, b) = (i_*(c), i_*(c))$ and we have exactness of

$$E_*(C) \xrightarrow{(i_*, i_*)} E_*(A) \oplus E_*(B) \xrightarrow{i_* \oplus -i_*} E_*(X).$$

Next, we prove exactness of

$$E_*(A) \oplus E_*(B) \xrightarrow{i_* \oplus -i_*} E_*(X) \xrightarrow{\delta} E_{*-1}(C)$$

The map δ is the composite of the bottom row

$$\begin{tikzcd}
 & E_*(A) \arrow[dl, "i_*"'] \arrow[dr, "\ell_*"] & & \\
E_*(X) \arrow[r, "j_*"] & E_*(X,B) \arrow[r, "\cong", "exc."'] & E_*(A,C) \arrow[r, "\partial"] & E_{*-1}(C).
\end{tikzcd}$$

Equivalently, δ can be computed as the composite of the bottom row

$$\begin{tikzcd}
 & E_*(B) \arrow[dl, "i_*"'] \arrow[dr, "\ell_*"] & & \\
E_*(X) \arrow[r, "j_*"] & E_*(X,A) \arrow[r, "\cong", "exc."'] & E_*(B,C) \arrow[r, "\partial"] & E_{*-1}(C).
\end{tikzcd}$$

So, that $\partial \circ (i_* \oplus -i_*) = 0$ is immediate.

If $\delta(x) = 0$, then $\partial(j_*(x)) = 0$. By exactness, of

$$E_*(A) \xrightarrow{\ell_*} E_*(A, C) \xrightarrow{\partial} E_{*-1}(C)$$

there is $a \in A$ such that $\ell_*(a) = j_*(x) \in E_*(A, C)$. That is

$$j_*(i_*(a)) = \ell_*(a) = j_*(x).$$

By exactness of

$$E_*(B) \xrightarrow{i_*} E_*(X) \xrightarrow{j_*} E_*(X, B)$$

$\ker(j_*) = i_*(E_*(B))$, so there is $b \in B$ such that

$$i_*(a) + i_*(b) = x.$$

Hence

$$(i_* \oplus -i_*)(a, -b) = x.$$

Finally, we check exactness at

$$E_*(X) \xrightarrow{\delta} E_{*-1}(C) \xrightarrow{(i_*, i_*)} E_{*-1}(A) \oplus E_{*-1}(B)$$

That the composite is zero follows from the definition of δ as either

$$E_*(X) \xrightarrow{j_*} E_*(X,B) \xrightarrow[exc.]{\cong} E_*(A,C) \xrightarrow{\partial} E_{*-1}(C) \xrightarrow{i_*} E_*(A)$$

or

$$E_*(X) \xrightarrow{j_*} E_*(X,A) \xrightarrow[exc.]{\cong} E_*(B,C) \xrightarrow{\partial} E_{*-1}(C) \xrightarrow{i_*} E_*(B).$$

Suppose that $(i_*(c), i_*(c)) = 0$. The following diagram commutes:

$$\begin{tikzcd}
E_*(X) \arrow[r, "j_*"] & E_*(X,B) \arrow[rr, "\cong", "exc."'] \arrow[dr, "\partial"'] & & E_*(A,C) \arrow[r, "\partial"] & E_{*-1}(C) \arrow[r, "i_*"] \arrow[dll, "i_*"] & E_{*-1}(A) \\
& & E_{*-1}(B) & & &
\end{tikzcd}$$

Hence, since $i_*(c) = 0$, there is $y \in E_*(X,B)$ such that $\partial(y) = c$. However, since $i_*(c) = 0 \in E_{*-1}(B)$, $\partial(y) = 0 \in E_{*-1}(B)$. It follows that there is $x \in E_*(X)$ such that $j_*(x) = y$. Hence, $\delta(x) = c$.

$\square$

3. Colimits

Exercise 3.1. Let $f_i : A_i \to A_{i+1}$ give a directed system of abelian groups:

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to \dots$$

Then for $s_i : A_i \to A_{i+1}$ given by $s_i(a) = a - f_i(a)$ and $s = \oplus s_i$, there is an exact sequence

$$0 \to \bigoplus_{i=1}^{\infty} A_i \xrightarrow{s} \bigoplus_{i=1}^{\infty} A_i \to \operatorname{colim} A_i \to 0.$$

That is, the colimit can be expressed as the coequalizer of the maps id and $\oplus f_i$.

Proof. We will fatten up the X_i in $\bigcup X_i$ in order to use Mayer-Vietoris.

Let

$$telX_i = \bigcup_{i=0}^{\infty} X_i \times [i, i+1]$$

where we identify

$$x_i \times \{i+1\} \in X_i \times [i, i+1] = j_i(x_i) \times \{i+1\} \in X_{i+1} \times [i+1, i+2].$$

Let $Y_k = \bigcup_{i=0}^{k-1} X_i \times [i, i+1] \cup (X_k \times \{k\})$. Then

$$telX_i = \operatorname{colim} Y_i$$

There are commutative diagrams:

$$\begin{tikzcd}
Y_0 \arrow[r, "r"] \arrow[d] & \dots \arrow[r] & Y_i \arrow[r] \arrow[d, "r"] & Y_{i+1} \arrow[r] \arrow[d, "r"] & \dots \\
X_0 \arrow[r] & \dots \arrow[r] & X_i \arrow[r, "j_i"] & X_{i+1} \arrow[r] & \dots
\end{tikzcd}$$

where the maps r are deformation retracts. Since $\pi_*(-)$ commutes with colimits, we have that the natural map

$$telX_i = \operatorname{colim} Y_i \to X$$

is a weak equivalence. Therefore,

$$E_*(telX_i) \to E_*(X)$$

is an isomorphism.

Let

$$A = (X_0 \times \{0\}) \cup \bigcup_{i \geq 0} (X_{2i} \times [2i + \frac{1}{2}, 2i+1]) \cup (X_{2i+1} \times [2i+1, 2i+2])$$

$$B = (X_0 \times [0, 1]) \cup \bigcup_{i \geq 1} (X_{2i-1} \times [2i - \frac{1}{2}, 2i]) \cup (X_{2i} \times [2i, 2i+1])$$

$$C = A \cap B = \bigcup_{i \geq 1} (X_i \cup X_i \times [i + 1/2, i+1]).$$

Then

$$A \xrightarrow{r} \bigcup_{i \geq 1} X_{2i-1} \qquad B \xrightarrow{r} \bigcup_{i \geq 0} X_{2i} \qquad C \xrightarrow{r} \bigcup_{i \geq 0} X_i.$$

Further, there is a commutative diagram

$$\begin{tikzcd} & C \arrow[r] \arrow[d, "r"'] & A \arrow[d, "r"] \\ \bigcup_{i\geq 0} X_i \arrow[r, equal] & \bigcup_{i\geq 0} X_{2i} \cup \bigcup_{i\geq 1} X_{2i-1} \arrow[r, "j_{2i-1}\cup \mathrm{id}_{X_{2i}}"] & \bigcup_{i\geq 1} X_{2i-1} \end{tikzcd}$$

and

$$\begin{tikzcd} & C \arrow[r] \arrow[d, "r"'] & B \arrow[d, "r"] \\ \bigcup_{i\geq 0} X_i \arrow[r, equal] & \bigcup_{i\geq 0} X_{2i} \cup \bigcup_{i\geq 1} X_{2i-1} \arrow[r, "\mathrm{id}_{X_{2i-1}}\cup j_{2i}"] & \bigcup_{i\geq 0} X_{2i} \end{tikzcd}$$

Now, note that $(telX_i; A, B)$ is an excisive triad. Let $k : A, B \to telX_i$ and $k_i : X_i \to X$ be the inclusions.

Using Mayer-Vietoris, we have a commutative diagram:

$$\begin{tikzcd} E_{*+1}(telX_i) \arrow[r] \arrow[d] & E_*(C) \arrow[r] \arrow[d, "\cong"] & E_*(A)\oplus E_*(B) \arrow[r, "k_*\oplus -k_*"] \arrow[d, "\cong"] & E_*(telX_i) \arrow[r] \arrow[d, "\cong"] & E_{*-1}(C) \\ 0 \arrow[r] & \bigoplus_{i\geq 0} E_*(X_i) \arrow[r, "\oplus_i(\mathrm{id}\oplus(j_i)_*)"] \arrow[d, "\oplus(-1)^i", "\cong"'] & \bigoplus_{i\geq 0} E_*(X_i) \arrow[r, "\oplus_{i\geq 0}(-1)^i(k_i)_*"] \arrow[d, "\oplus(-1)^i", "\cong"'] & E_*(X) \arrow[r] & 0 \arrow[u] \\ 0 \arrow[r] & \bigoplus_{i\geq 0} E_*(X_i) \arrow[r, "\oplus_i(\mathrm{id}\oplus(-1)(j_i)_*)"] & \bigoplus_{i\geq 0} E_*(X_i) \arrow[r] & \operatorname{colim} E_*(X_i) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

which establishes the isomorphism. $\square$