

MATH 6280 - CLASS 32

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These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. PREVIEW

Let $X = \bigcup_{i=0}^{\infty} X_i$ for $j_i : X_i \subseteq X_{i+1}$. Recall that since S^n and $S^n \times I$ are compact, we have

$$\operatorname{colim} \pi_*(X_i) \rightarrow \pi_*(X)$$

The next goal is to prove that this holds for generalized homology theories as well.

Theorem 1.1. *Let $E_* : \mathbf{Toppairs} \rightarrow \mathbf{Ab}$ be a generalized homology theory, then the natural map*

$$\operatorname{colim} E_*(X_i) \rightarrow E_*(X)$$

is an isomorphism.

We start with some remarks.

Remark 1.2. Recall that a coequalizer is the colimit of a diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

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or equivalently, the pushout:

$$\begin{array}{ccc} A \amalg A & \xrightarrow{f \sqcup g} & B \\ \nabla \downarrow & & \\ A & & \end{array}$$

Let \mathcal{A} be a category with coproducts and co-equalizers. For $f_i : A_i \rightarrow A_{i+1}$ a directed system, $\text{colim } A_i$ is the coequalizer:

$$\amalg A_i \xrightleftharpoons[\sqcup f_i]{\sqcup id} \amalg A_i \longrightarrow \text{colim } A_i$$

(In abelian groups, these diagrams can be rewritten as an exact sequence

$$0 \rightarrow \bigoplus_{i=1}^{\infty} A_i \xrightarrow{s} \bigoplus_{i=1}^{\infty} A_i \rightarrow \text{colim } A_i \rightarrow 0$$

for $s_i : A_i \rightarrow A_{i+1}$ given by $s_i(a) = a - f_i(a)$ and $s = \oplus s_i$.)

Therefore, if a functor preserves arbitrary coproducts and coequalizers, then it preserves directed colimits. Generalized homology theories do preserve coproducts, but we need some kind of approximation of what it would be for it to preserve co-equalizers, or at least, how it behaves with pushouts. This is what Mayer-Vietoris will give us.

2. MAYER-VIETORIS

There are many versions of this theorem. I'll give a simple one and let you figure out others.

Theorem 2.1. *Let $E_* : \text{CWTop pairs} \rightarrow \text{Ab}$. Let $(X; A, B)$ be a CW-triad and i denote the various inclusions. Let $C = A \cap B$.*

Then there is a long exact sequence

$$\dots \rightarrow E_*(C) \xrightarrow{(i_*, i_*)} E_*(A) \oplus E_*(B) \xrightarrow{i_* \oplus -i_*} E_*(X) \xrightarrow{\delta} E_{*-1}(C) \rightarrow \dots$$

where δ is the composit:

$$E_*(X) \longrightarrow E_*(X, B) \xrightarrow[\text{exc.}]{\cong} E_*(A, C) \xrightarrow{\partial} E_{*-1}(C).$$

Remark 2.2. Under these conditions, we have that

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & X \end{array}$$

is a (homotopy) pushout and MVT says that $E_*(-)$ turns this into a long exact sequence.

First, note the following lemma.

Lemma 2.3. *Let $E_* : \text{CWTop pairs} \rightarrow \text{Ab}$. Let $(X; A, B)$ be a CW-triad and i denote the various inclusions. Let $C = A \cap B$. Then*

$$E_*(A, C) \oplus E_*(B, C) \rightarrow E_*(X, C)$$

is an isomorphism. Further, the inclusion of the factors is split by the excision isomorphism:

$$E_*(A, C) \longrightarrow E_*(X, C) \longrightarrow E_*(X, B) \xrightarrow[\text{exc.}]{\cong} E_*(A, C) ,$$

and equivalently,

$$E_*(X, C) \xrightarrow{i_* \oplus i_*} E_*(X, A) \oplus E_*(X, B)$$

is an isomorphism.

Proof. We have $X/C \simeq A/C \vee B/C$. Then

$$\begin{aligned} E_*(A, C) \oplus E_*(B, C) &= \tilde{E}_*(A/C) \oplus \tilde{E}_*(B/C) \\ &\xrightarrow{i_* \oplus i_*} \tilde{E}_*(A/C \vee B/C) \\ &\cong \tilde{E}_*(X/C) \\ &= E_*(X, C). \end{aligned}$$

For CW triad, the excision isomorphism comes from the equivalence $X/B \cong A/C$ and the second claim follows. \square

Proof of Theorem 2.1. As in *Concise*, the following diagram commutes:

$$\begin{array}{ccccccc} & & E_*(A) & & E_*(X, B) & \xleftarrow{\cong} & E_*(A, C) \\ & \nearrow & & \searrow & \nearrow & & \searrow \\ E_*(C) & \longrightarrow & E_*(X) & \xrightarrow{k_*} & E_*(X, C) & \xrightarrow{\partial} & E_{*-1}(C) \\ & \searrow & \nearrow & \searrow & \nearrow & & \nearrow \\ & & E_*(B) & & E_*(X, A) & \xleftarrow{\cong} & E_*(B, C) \end{array}$$

We use the notation $a \in A$, $b \in B$, $c \in C$ and $x \in X$

We first prove exactness at

$$E_*(C) \xrightarrow{(i_*, i_*)} E_*(A) \oplus E_*(B) \xrightarrow{i_* \oplus -i_*} E_*(X).$$

That the composition is zero is obvious. Suppose $i_*(a) - i_*(b) = 0$. Then $i_*(a) = i_*(b)$. Then, by exactness of

$$E_*(A) \rightarrow E_*(X) \xrightarrow{j_*} E_*(X, A)$$

we have $j_*(i_*(a)) = 0 \in E_*(X, A)$ and similarly, $j_*(i_*(b)) = 0 \in E_*(X, B)$.

Since $E_*(X, C) \cong E_*(X, A) \oplus E_*(X, B)$

$$k_*(i_*(a)) = k_*(i_*(b)) = 0.$$

There is a $c \in E_*(C)$ such that $i_*(c) = i_*(a) = i_*(b) \in E_*(X)$. Hence, $(a, b) = (i_*(c), i_*(c))$ and we have exactness of

$$E_*(C) \xrightarrow{(i_*, i_*)} E_*(A) \oplus E_*(B) \xrightarrow{i_* \oplus -i_*} E_*(X).$$

Next, we prove exactness of

$$E_*(A) \oplus E_*(B) \xrightarrow{i_* \oplus -i_*} E_*(X) \xrightarrow{\delta} E_{*-1}(C)$$

The map δ is the composite of the bottom row

$$\begin{array}{ccccccc} & & E_*(A) & & & & \\ & \swarrow i_* & & \searrow \ell_* & & & \\ E_*(X) & \xrightarrow{j_*} & E_*(X, B) & \xrightarrow[\text{exc.}]{\cong} & E_*(A, C) & \xrightarrow{\partial} & E_{*-1}(C). \end{array}$$

Equivalently, δ can be computed as the composite of the bottom row

$$\begin{array}{ccccccc} & & E_*(B) & & & & \\ & \swarrow i_* & & \searrow \ell_* & & & \\ E_*(X) & \xrightarrow{j_*} & E_*(X, A) & \xrightarrow[\text{exc.}]{\cong} & E_*(B, C) & \xrightarrow{\partial} & E_{*-1}(C). \end{array}$$

So, that $\partial \circ (i_* \oplus -i_*) = 0$ is immediate.

If $\delta(x) = 0$, then $\partial(j_*(x)) = 0$. By exactness, of

$$E_*(A) \xrightarrow{\ell_*} E_*(A, C) \xrightarrow{\partial} E_{*-1}(C)$$

there is $a \in A$ such that $\ell_*(a) = j_*(x) \in E_*(A, C)$. That is

$$j_*(i_*(a)) = \ell_*(a) = j_*(x).$$

By exactness of

$$E_*(B) \xrightarrow{i_*} E_*(X) \xrightarrow{j_*} E_*(X, B)$$

$\ker(j_*) = i_*(E_*(B))$, so there is $b \in B$ such that

$$i_*(a) + i_*(b) = x.$$

Hence

$$(i_* \oplus -i_*)(a, -b) = x.$$

Finally, we check exactness at

$$E_*(X) \xrightarrow{\delta} E_{*-1}(C) \xrightarrow{(i_*, i_*)} E_{*-1}(A) \oplus E_{*-1}(B)$$

That the composite is zero follows from the definition of δ as either

$$E_*(X) \xrightarrow{j_*} E_*(X, B) \xrightarrow[\text{exc.}]{\cong} E_*(A, C) \xrightarrow{\partial} E_{*-1}(C) \xrightarrow{i_*} E_*(A)$$

or

$$E_*(X) \xrightarrow{j_*} E_*(X, A) \xrightarrow[\text{exc.}]{\cong} E_*(B, C) \xrightarrow{\partial} E_{*-1}(C) \xrightarrow{i_*} E_*(B).$$

Suppose that $(i_*(c), i_*(c)) = 0$. The following diagram commutes:

$$\begin{array}{ccccccc} E_*(X) & \xrightarrow{j_*} & E_*(X, B) & \xrightarrow[\text{exc.}]{\cong} & E_*(A, C) & \xrightarrow{\partial} & E_{*-1}(C) \xrightarrow{i_*} E_{*-1}(A) \\ & & \searrow \partial & & \swarrow i_* & & \\ & & & E_{*-1}(B) & & & \end{array}$$

Hence, since $i_*(c) = 0$, there is $y \in E_*(X, B)$ such that $\partial(y) = c$. However, since $i_*(c) = 0 \in E_{*-1}(B)$, $\partial(y) = 0 \in E_{*-1}(B)$. It follows that there is $x \in E_*(X)$ such that $j_*(x) = y$. Hence, $\delta(x) = c$.

□

3. COLIMITS

Exercise 3.1. Let $f_i : A_i \rightarrow A_{i+1}$ give a directed system of abelian groups:

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \rightarrow \dots$$

Then for $s_i : A_i \rightarrow A_{i+1}$ given by $s_i(a) = a - f_i(a)$ and $s = \oplus s_i$, there is an exact sequence

$$0 \rightarrow \bigoplus_{i=1}^{\infty} A_i \xrightarrow{s} \bigoplus_{i=1}^{\infty} A_i \rightarrow \text{colim } A_i \rightarrow 0.$$

That is, the colimit can be expressed as the coequalizer of the maps id and $\oplus f_i$.

Proof. We will fatten up the X_i in $\bigcup X_i$ in order to use Mayer-Vietoris.

Let

$$telX_i = \bigcup_{i=0}^{\infty} X_i \times [i, i+1]$$

where we identify

$$x_i \times \{i+1\} \in X_i \times [i, i+1] = j_i(x_i) \times \{i+1\} \in X_{i+1} \times [i+1, i+2].$$

Let $Y_k = \bigcup_{i=0}^{k-1} X_i \times [i, i+1] \cup (X_k \times \{k\})$. Then

$$telX_i = \text{colim } Y_i$$

There are commutative diagrams:

$$\begin{array}{ccccccc} Y_0 & \xrightarrow{r} & \dots & \longrightarrow & Y_i & \longrightarrow & Y_{i+1} \longrightarrow \dots \\ \downarrow & & & & \downarrow r & & \downarrow r \\ X_0 & \longrightarrow & \dots & \longrightarrow & X_i & \xrightarrow{j_i} & X_{i+1} \longrightarrow \dots \end{array}$$

where the maps r are deformation retracts. Since $\pi_*(-)$ commutes with colimits, we have that the natural map

$$telX_i = \text{colim } Y_i \rightarrow X$$

is a weak equivalence. Therefore,

$$E_*(telX_i) \rightarrow E_*(X)$$

is an isomorphism.

Let

$$A = (X_0 \times \{0\}) \cup \bigcup_{i \geq 0} (X_{2i} \times [2i + \frac{1}{2}, 2i+1]) \cup (X_{2i+1} \times [2i+1, 2i+2])$$

$$B = (X_0 \times [0, 1]) \cup \bigcup_{i \geq 1} (X_{2i-1} \times [2i - \frac{1}{2}, 2i]) \cup (X_{2i} \times [2i, 2i+1])$$

$$C = A \cap B = \bigcup_{i \geq 1} (X_i \cup X_i \times [i + 1/2, i+1]).$$

Then

$$A \xrightarrow{r} \bigcup_{i \geq 1} X_{2i-1} \qquad B \xrightarrow{r} \bigcup_{i \geq 0} X_{2i} \qquad C \xrightarrow{r} \bigcup_{i \geq 0} X_i.$$

Further, there is a commutative diagram

$$\begin{array}{ccc}
 C & \xrightarrow{\quad} & A \\
 r \downarrow & & \downarrow r \\
 \bigcup_{i \geq 0} X_i & \xrightarrow{j_{2i-1} \cup \text{id}_{X_{2i}}} & \bigcup_{i \geq 1} X_{2i-1}
 \end{array}$$

$\bigcup_{i \geq 0} X_i = \bigcup_{i \geq 0} X_{2i} \cup \bigcup_{i \geq 1} X_{2i-1}$

and

$$\begin{array}{ccc}
 C & \xrightarrow{\quad} & B \\
 r \downarrow & & \downarrow r \\
 \bigcup_{i \geq 0} X_i & \xrightarrow{\text{id}_{X_{2i-1}} \cup j_{2i}} & \bigcup_{i \geq 0} X_{2i}
 \end{array}$$

$\bigcup_{i \geq 0} X_i = \bigcup_{i \geq 0} X_{2i} \cup \bigcup_{i \geq 1} X_{2i-1}$

Now, note that $(\text{tel} X_i; A, B)$ is an excisive triad. Let $k : A, B \rightarrow \text{tel} X_i$ and $k_i : X_i \rightarrow X$ be the inclusions.

Using Mayer-Vietoris, we have a commutative diagram:

$$\begin{array}{ccccccc}
 E_{*+1}(\text{tel} X_i) & \longrightarrow & E_*(C) & \longrightarrow & E_*(A) \oplus E_*(B) & \xrightarrow{k_* \oplus -k_*} & E_*(\text{tel} X_i) \longrightarrow E_{*-1}(C) \\
 \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & \bigoplus_{i \geq 0} E_*(X_i) & \xrightarrow{\oplus_i (\text{id} \oplus (j_i)_*)} & \bigoplus_{i \geq 0} E_*(X_i) & \xrightarrow{\oplus_{i \geq 0} (-1)^i (k_i)_*} & E_*(X) \longrightarrow 0 \\
 & & \cong \downarrow \oplus (-1)^i & & \cong \downarrow \oplus (-1)^i & & \uparrow \\
 0 & \longrightarrow & \bigoplus_{i \geq 0} E_*(X_i) & \xrightarrow{\oplus_i (\text{id} \oplus (-1)(j_i)_*)} & \bigoplus_{i \geq 0} E_*(X_i) & \longrightarrow & \text{colim } E_*(X_i) \longrightarrow 0
 \end{array}$$

which establishes the isomorphism. □