

MATH 6280 - CLASS 3

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1. NATURAL TRANSFORMATION

Definition 1.1. A *natural transformation* is a morphism of functors. That is, if $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation $\eta : F \rightarrow G$ is a collection of morphisms $\eta_X : F(X) \rightarrow G(X)$ which make the following diagrams commute for every $f : X \rightarrow Y$ in \mathcal{C} :

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow \eta_X & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

If each η_X is an isomorphism, then η is a *natural isomorphism* and we write $F \cong G$.

Example 1.2. (1) Let A and B be abelian groups. There are functors

$$\text{Hom}(A \otimes B, -) : \text{Ab} \rightarrow \text{Ab}$$

and

$$\text{Hom}(A, \text{Hom}(B, -)) : \text{Ab} \rightarrow \text{Ab}.$$

Further, there is a map

$$\eta_C : \text{Hom}(A \otimes B, C) \rightarrow \text{Hom}(A, \text{Hom}(B, C))$$

defined by

$$\eta_C(f)(a)(b) = f(a \otimes b).$$

Then η defines a natural transformation:

$$\eta : \text{Hom}(A \otimes B, -) \rightarrow \text{Hom}(A, \text{Hom}(B, -))$$

which is in fact a natural isomorphism.

- (2) If $F : \mathcal{C} \rightarrow \text{Sets}$ is representable, this means that there exists $X \in \mathcal{C}$ and a natural isomorphism $F(-) \rightarrow \mathcal{C}(X, -)$.
- (3) The Hurewicz homomorphism:

$$\pi_1(X) \rightarrow H_1(X)$$

will be an example of a natural transformation.

- (4) If $F : \text{Ab} \rightarrow \text{Top}$ is the free abelian group functor and $U : \text{Ab} \rightarrow \text{Sets}$ is the forgetful functor, then there are natural transformations $F \circ U \rightarrow \text{id}_{\text{Ab}}$ and $\text{id}_{\text{Sets}} \rightarrow U \circ F$.

Remark 1.3. Whenever we say that two things are *naturally* isomorphic, it means that there are functors lying around and a natural isomorphism between them.

Exercise 1.4. A natural transformation is a kind of categorical homotopy. Let I be the category:

$$\begin{array}{ccc} \curvearrowright & & \curvearrowright \\ 0 & \longrightarrow & 1 \end{array}$$

and $F, G : \mathcal{C} \rightarrow \mathcal{D}$. Then a natural transformation is equivalent to a functor:

$$\eta : \mathcal{C} \times I \rightarrow \mathcal{D}$$

that satisfies $\eta(X, 0) = F(X)$ and $\eta(X, 1) = G(X)$.

Definition 1.5.

- Categories \mathcal{C} and \mathcal{D} are *isomorphic* if there exists functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G = \text{id}_{\mathcal{D}}$ and $G \circ F = \text{id}_{\mathcal{C}}$.
- Categories \mathcal{C} and \mathcal{D} are *equivalent* if there exists functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms $F \circ G \cong \text{id}_{\mathcal{D}}$ and $G \circ F \cong \text{id}_{\mathcal{C}}$.

2. ADJUNCTIONS

Definition 2.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$. Then we say that F and G are an adjoint pair if there is a natural isomorphism

$$\mathcal{C}(F(X), Y) \cong \mathcal{D}(X, G(Y)).$$

In this case, we say that F is the left adjoint and G is the right adjoint. Finally, we write

$$F : \mathcal{C} \overset{\perp}{\rightleftarrows} \mathcal{D} : G$$

Example 2.2. • Consider the functors $U : \text{Ab} \rightarrow \text{Sets}$ which sends A to the set underlying A and $F : \text{Sets} \rightarrow \text{Ab}$ which sends a set S to the free abelian group $F(S)$ generated by S . Note that

$$\text{Ab}(F(X), Y) \cong \text{Ab}(X, U(Y))$$

and that this isomorphism is natural in each variables. This is an example of an adjunction.

$$F : \text{Sets} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} \text{Ab} : U$$

• Fix a set Y . There are functors $- \times Y : \text{Sets} \rightarrow \text{Sets}$ and $\text{Hom}(Y, -) : \text{Sets} \rightarrow \text{Sets}$ and these give rise to an adjunction:

$$- \times Y : \text{Sets} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} \text{Sets} : \text{Hom}(Y, -)$$

which just reflex the isomorphism:

$$\text{Sets}(X \times Y, Z) \cong \text{Sets}(X, \text{Sets}(Y, Z)).$$

• In abelian groups, you get a similar adjunction:

$$- \otimes B : \text{Ab} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} \text{Ab} : \text{Hom}(B, -)$$

which just reflex the isomorphism:

$$\text{Ab}(A \otimes B, C) \cong \text{Ab}(A, \text{Hom}(B, C)).$$

and these kinds of adjunctions are often called *tensor-hom* adjunctions.

3. YONEDA

Lemma 3.1 (Yoneda). *Given $F : \mathcal{C} \rightarrow \text{Sets}$ and $A \in \mathcal{C}$, the natural transformations $\mathcal{C}(A, -) \rightarrow F$ are in bijective correspondence with the elements of $F(A)$.*

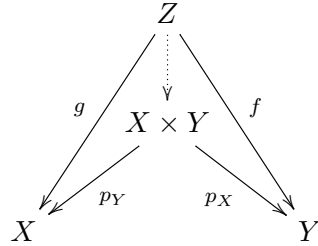
Proof sketch. Let $a \in F(A)$. Then

$$\begin{aligned} \eta_X : \mathcal{C}(A, X) &\rightarrow F(X) \\ f &\mapsto F(f)(a) \end{aligned}$$

is a natural transformation. Conversely, if $\eta : \mathcal{C}(A, -) \rightarrow F$ is a natural transformation, let $a = \eta_A(\text{id}_A)$ and check that η_X must be as above for this a . □

4. LIMITS AND COLIMITS

Example 4.1 (Limit: product). The product of $X \times Y$ is an object of \mathcal{C} with the following property:

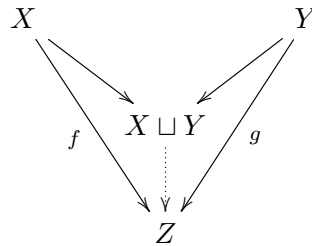


Given maps $f : Z \rightarrow X$ and $g : Z \rightarrow Y$, there exists a unique map $f \times g : Z \rightarrow X \times Y$ making the above diagram commute.

Depending on \mathcal{C} , the product map or may not exist.

- The product in Sets, Top, Gr and Ab is just the cartesian product.
- The product in Top_* is $(X, *) \times (Y, *) = (X \times Y, (*, *))$. **Warning:** This is not the smash product. Remember, the smash product will not be the categorical product in Top_* .

Example 4.2 (Colimit: coproduct). The coproduct of $X \sqcup Y$ is an object of \mathcal{C} with the following property:



Given maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$, there exists a unique map $f \sqcup g : X \sqcup Y \rightarrow Z$ making the above diagram commute.

Depending on \mathcal{C} , the coproduct map or may not exist.

- The coproduct in Sets, Top is the disjoint union.
- The coproduct in Top_* is the wedge $X \vee Y$.
- The coproduct in Ab is direct sum, $X \sqcup Y = X \oplus Y$.

Details. Given two maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$. Then, one can define a map

$$f \sqcup g = f + g : X \oplus Y \rightarrow Z$$

by $(f + g)((a, b)) = f(a) + g(b)$. Further, suppose that $F : X \oplus Y \rightarrow Z$ is a map, then we get maps

$$f(a) = F((a, 0)), \quad g(b) = F((0, b)).$$

These correspondences are inverse to one another.

- The coproduct in Gr is the free product $*$.

Example 4.3 (Limit: pull-back). Consider a diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow i \\ Y & \xrightarrow{j} & Z \end{array}$$

The pull-back of the diagram is an element $X \times_Z Y$ of \mathcal{C} with maps $X \times_Z Y \rightarrow X$, $X \times_Z Y \rightarrow Y$ such that, given maps $f : W \rightarrow X$ and $g : W \rightarrow Y$ such that $if = jg$, there exists a unique map $W \rightarrow X \times_Z Y$ making the following diagram commute:

$$\begin{array}{ccccc} W & & & & \\ & \searrow f & & & \\ & & X \times_Z Y & \xrightarrow{\quad} & X \\ & \swarrow g & \downarrow & & \downarrow i \\ & & Y & \xrightarrow{j} & Z \end{array}$$

- In Sets, Ab, Gr, Top

$$X \times_Z Y = \{(x, y) \mid i(x) = j(y)\} \subseteq X \times Y$$

Details. If $f : W \rightarrow X$ and $g : W \rightarrow Z$, then $f \times g : W \rightarrow X \times_Z Y$ is given by

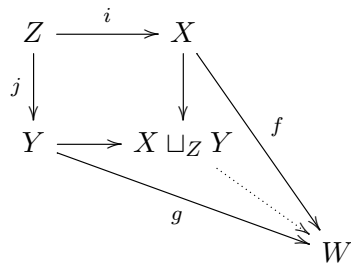
$$(f \times g)(z) = (f(z), g(z)).$$

Example 4.4 (Colimit: push-out). Consider a diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ j \downarrow & & \\ & & Y \end{array}$$

The push-out of the diagram is an element $X \sqcup_Z Y$ of \mathcal{C} with maps $X \rightarrow X \sqcup_Z Y$, $Y \rightarrow X \sqcup_Z Y$ such that, given maps $f : X \rightarrow W$ and $g : Y \rightarrow W$ such that $fi = gj$, there exists a unique map

$X \sqcup_Z Y \rightarrow W$ making the following diagram commute:



- In Sets and Top,

$$X \sqcup_Z Y = (X \sqcup Y) / (j(z) \sim i(z))$$

Exercise 4.5. What is the pushout in groups, abelian groups and commutative rings?