MATH 6280 - CLASS 3

Contents

1. Natural Transformation

Definition 1.1. A *natural transformation* is a morphism of functors. That is, if $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\eta : F \to G$ is a collection of morphisms $\eta_X : F(X) \to G(X)$ which make the following diagrams commute for every $f : X \to Y$ in $\mathcal{C}$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y). \end{tikzcd}$$

If each η_X is an isomorphism, then η is a *natural isomorphism* and we write $F \cong G$.

Example 1.2. (1) Let A and B be abelian groups. There are functors

$$\mathrm{Hom}(A \otimes B, -) : \mathrm{Ab} \to \mathrm{Ab}$$

and

$$\mathrm{Hom}(A, \mathrm{Hom}(B, -)) : \mathrm{Ab} \to \mathrm{Ab}\,.$$

Further, there is a map

$$\eta_C : \mathrm{Hom}(A \otimes B, C) \to \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

defined by

$$\eta_C(f)(a)(b) = f(a \otimes b).$$

Then η defines a natural transformation:

$$\eta : \mathrm{Hom}(A \otimes B, -) \to \mathrm{Hom}(A, \mathrm{Hom}(B, -))$$

which is in fact a natural isomorphism.

(2) If $F : \mathcal{C} \to \mathrm{Sets}$ is representable, this means that there exists $X \in \mathcal{C}$ and a natural isomorphism $F(-) \to \mathcal{C}(X, -)$.

(3) The Hurewicz homomorphism:

$$\pi_1(X) \to H_1(X)$$

will be an example of a natural transformation.

(4) If $F : \mathrm{Ab} \to \mathrm{Top}$ is the free abelian group functor and $U : \mathrm{Ab} \to \mathrm{Sets}$ is the forgetful functor, then there are natural transformations $F \circ U \to \mathrm{id}_{Ab}$ and $\mathrm{id}_{\mathrm{Sets}} \to U \circ F$.

Remark 1.3. Whenever we say that two things are *naturally* isomorphic, it means that there are functors lying around and a natural isomorphism between them.

Exercise 1.4. A natural transformation is a kind of categorical homotopy. Let I be the category:

$$\begin{tikzcd} 0 \arrow[loop above] \arrow[r] & 1 \arrow[loop above] \end{tikzcd}$$

and $F, G : \mathcal{C} \to \mathcal{D}$. Then a natural transformation is equivalent to a functor:

$$\eta : \mathcal{C} \times I \to \mathcal{D}$$

that satisfies $\eta(X, 0) = F(X)$ and $\eta(X, 1) = G(X)$.

Definition 1.5.
- Categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$.
- Categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ f \cong \mathrm{id}_{\mathcal{D}}$.

2. Adjunctions

Definition 2.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Then we say that F and G are an adjoint pair if there is a natural isomorphism

$$\mathcal{C}(F(X), Y) \cong \mathcal{D}(X, G(Y)).$$

In this case, we say that F is the left adjoint and G is the right adjoint. Finally, we write

$$\begin{tikzcd} F : \mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{D} : G \arrow[l, shift left] \end{tikzcd}$$

Example 2.2.

- Consider the functors $U : \mathrm{Ab} \to \mathrm{Sets}$ which sends A to the set underlying A and $F : \mathrm{Sets} \to \mathrm{Ab}$ which sends a set S to the free abelian group $F(S)$ generated by S. Note that
$$\mathrm{Ab}(F(X), Y) \cong \mathrm{Ab}(X, U(Y))$$
and that this isomorphism is natural in each variables. This is an example of an adjunction.
$$F : \mathrm{Sets} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Ab} : U$$
- Fix a set Y. There are functors $- \times Y : \mathrm{Sets} \to \mathrm{Sets}$ an $\mathrm{Hom}(Y, -)$ and these give rise to an adjunction:
$$- \times Y : \mathrm{Sets} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Sets} : \mathrm{Hom}(Y, -)$$
which just reflex the isomorphism:
$$\mathrm{Sets}(X \times Y, Z) \cong \mathrm{Sets}(X, \mathrm{Sets}(Y, Z)).$$
- In abelian groups, you get a similar adjunction:
$$- \otimes B : \mathrm{Ab} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Ab} : \mathrm{Hom}(B, -)$$
which just reflex the isomorphism:
$$\mathrm{Ab}(A \otimes B, C) \cong \mathrm{Ab}(A, \mathrm{Hom}(B, C)).$$
and these kinds of adjunctions are often called *tensor-hom* adjunctions.

3. Yoneda

Lemma 3.1 (Yoneda)**.** *Given $F : \mathcal{C} \to \mathrm{Sets}$ and $A \in \mathcal{C}$, the natural transformations $\mathcal{C}(A, -) \to F$ are in bijective correspondence with the elements of $F(A)$.*

Proof sketch. Let $a \in F(A)$. Then
$$\begin{aligned} \eta_X : \mathcal{C}(A, X) &\to F(X) \\ f &\mapsto F(f)(a) \end{aligned}$$
is a natural transformation. Conversely, if $\eta : \mathcal{C}(A, -) \to F$ is a natural transformation, let $a = \eta_A(\mathrm{id}_A)$ and check that η_X must be as above for this a. □

4. Limits and Colimits

Example 4.1 (Limit: product)**.** The product of $X \times Y$ is an object of $\mathcal{C}$ with the following property:

$$\begin{tikzcd} & Z \arrow[ddl, "g"'] \arrow[ddr, "f"] \arrow[d, dotted] & \\ & X \times Y \arrow[dl, "p_Y"] \arrow[dr, "p_X"'] & \\ X & & Y \end{tikzcd}$$

Given maps $f : Z \to X$ and $g : Z \to Y$, there exists a unique map $f \times g : Z \to X \times Y$ making the above diagram commute.

Depending on $\mathcal{C}$, the product map or may not exist.

- The product in Sets, Top, Gr and Ab is just the cartesian product.
- The product in Top_* is $(X, *) \times (Y, *) = (X \times Y, (*, *))$. **Warning**: This is not the smash product. Remember, the smash product will not be the categorical product in Top_*.

Example 4.2 (Colimit: coproduct)**.** The coproduct of $X \sqcup Y$ is an object of $\mathcal{C}$ with the following property:

$$\begin{tikzcd} X \arrow[dr] \arrow[ddr, "f"'] & & Y \arrow[dl] \arrow[ddl, "g"] \\ & X \sqcup Y \arrow[d, dotted] & \\ & Z & \end{tikzcd}$$

Given maps $f : X \to Z$ and $g : Y \to Z$, there exists a unique map $f \sqcup g : X \sqcup Y \to Z$ making the above diagram commute.

Depending on $\mathcal{C}$, the coproduct map or may not exist.

- The coproduct in Sets, Top is the disjoint union.
- The coproduct in Top_* is the wedge $X \vee Y$.
- The coproduct in Ab is direct sum, $X \sqcup Y = X \oplus Y$.

Details. *Given two maps $f : X \to Z$ and $g : Y \to Z$. Then, one can define a map*

$$f \sqcup g = f + g : X \oplus Y \to Z$$

by $(f+g)((a,b)) = f(a)+g(b)$. Further, suppose that $F : X \oplus Y \to Z$ is a map, then we get maps

$$f(a) = F((a,0)), \qquad g(b) = F((0,b)).$$

These correspondences are inverse to one another.

- The coproduct in Gr is the free product $*$.

Example 4.3 (Limit: pull-back)**.** Consider a diagram

$$\begin{tikzcd} & X \arrow[d, "i"] \\ Y \arrow[r, "j"'] & Z \end{tikzcd}$$

The pull-back of the diagram is an element $X \times_Z Y$ of $\mathcal{C}$ with maps $X \times_Z Y \to X$, $X \times_Z Y \to Y$ such that, given maps $f : W \to X$ and $g : W \to Y$ such that $if = jg$, there exists a unique map $W \to X \times_Z Y$ making the following diagram commute:

$$\begin{tikzcd} W \arrow[dr, dotted] \arrow[drr, "f"] \arrow[ddr, "g"'] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "i"] \\ & Y \arrow[r, "j"'] & Z \end{tikzcd}$$

- In Sets, Ab, Gr, Top

$$X \times_Z Y = \{(x,y) \mid i(x) = j(y)\} \subseteq X \times Y$$

Details. *If $f : W \to X$ and $g : W \to Z$, then $f \times g : W \to X \times_Z Y$ is given by*

$$(f \times g)(z) = (f(z), g(z)).$$

Example 4.4 (Colimit: push-out)**.** Consider a diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "j"'] & X \\ Y & \end{tikzcd}$$

The push-out of the diagram is an element $X \sqcup_Z Y$ of $\mathcal{C}$ with maps $X \to X \sqcup_Z Y$, $Y \to X \sqcup_Z Y$ such that, given maps $f : X \to W$ and $g : Y \to W$ such that $fi = gj$, there exists a unique map

$X \sqcup_Z Y \to W$ making the following diagram commute:

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d, "j"'] & X \arrow[d] \arrow[ddr, "f"] & \\
Y \arrow[r] \arrow[drr, "g"'] & X \sqcup_Z Y \arrow[dr, dotted] & \\
& & W
\end{tikzcd}$$

- In Sets and Top,

$$X \sqcup_Z Y = (X \sqcup Y)/(j(z) \sim i(z))$$

Exercise 4.5. What is the pushout in groups, abelian groups and commutative rings?