MATH 6280 - CLASS 28

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

Remark 0.1 (Passage from reduced to unreduced)**.** Suppose that we are given an unreduced homology theory E. Let X pointed CW-complex. Note that

$$* \to X \to *$$

is the identity. This implies that the long exact sequence

$$\ldots \to E_{q+1}(X,*) \to E_q(*) \to E_q(X) \to E_q(X,*) \to E_{q-1}(*) \to \ldots$$

splits into split short exact sequences

$$0 \to E_q(*) \to E_q(X) \to E_q(X,*) \to 0$$

so that

$$E_*(X) \cong E_*(X,*) \oplus E_*(*).$$

Then, the functor

$$\widetilde{E}_*(X) = E_*(X,*)$$

gives a reduced homology theory.

Conversely, given a reduced homology theory $\widetilde{E}_*$ and a CW-pair (X, A). Then letting

$$E_*(X, A) = \widetilde{E}_*(X_+/A_+).$$

In particular,

$$E_*(X) = E_*(X, \emptyset) = \widetilde{E}_*(X_+/\emptyset_+) = \widetilde{E}_*(X_+).$$

This will give an unreduced homology theory.

Note that nothing in this remark appealed to the dimension axiom.

Theorem 0.2. *The following data is equivalent:*

(1) $E_* : \mathrm{Toppairs} \to \mathrm{Ab}$
(2) $E_* : \mathrm{CWpairs} \to \mathrm{Ab}$
(3) $\widetilde{E}_* : \mathrm{CWTop}_* \to \mathrm{Ab}$
(4) $\widetilde{E}_* : \mathrm{Top}_* \to \mathrm{Ab}$

The same holds for cohomology.

1. Uniqueness

Definition 1.1. An isomorphism $\alpha : \widetilde{E}_* \to \widetilde{E}'_*$ of reduced cohomology theories is a natural isomorphism that commutes with the suspension isomorphisms, i.e.,

$$\begin{tikzcd}
\widetilde{E}_n(X) \arrow[r, "\alpha_X"] \arrow[d, "\Sigma"'] & \widetilde{E}'_n(\Sigma X) \arrow[d, "\Sigma"] \\
\widetilde{E}_{n+1}(\Sigma X) \arrow[r, "\alpha_{\Sigma X}"] & \widetilde{E}'_{n+1}(\Sigma X).
\end{tikzcd}$$

In this section, we are going to assume what Katharyn and Andy proved las class and prove the uniqueness of $\widetilde{H}$.

Theorem 1.2. *Let $\widetilde{E}$ is be a reduced cohomology theory such that $\widetilde{E}_*(S^0) = \mathbb{Z}$, then $\widetilde{E} \cong \widetilde{H}$.*

Note that this also holds for coefficients other than $\mathbb{Z}$. Again, this is exactly the argument in *Concise*.

Proof. Assume $\widetilde{E} : \mathrm{CWTop}_* \to \mathrm{Ab}$ is a homology theory that satisfies the dimension axiom.

We will assume the following fact and prove it later.

Theorem 1.3. *Let E be any generalized homology theory. Let $X = \bigcup_{i=1}^{\infty} X_i$ where $X_0 \subset X_1 \subset X_2 \subset \ldots$. Then $E_*(X) = \operatorname{colim} E_*(X_i)$.*

As Katharyn and Andy showed, this implies that,

- $\widetilde{E}_n(X) = \widetilde{E}_n(X^{n+1})$ for any X.
- $\pi_n^{ab}(\bigvee S^n) \xrightarrow{h} \widetilde{E}_n(\bigvee S^n)$ is an isomorphism.

We use $\widetilde{E}$ to define the cellular chain complex. Let $\widetilde{C}_n^E(X) = \widetilde{E}_n(X^n/X^{n-1})$. Then $\widetilde{C}_n^E(X) \cong \pi_n^{ab}(X^n/X^{n-1})$. So, as abelian groups

$$\widetilde{C}_n^E(X) \cong \widetilde{C}_n^{CW}(X).$$

Define $d_n : \widetilde{C}_n^E(X) \to \widetilde{C}_{n-1}^E(X)$ by $\Sigma^{-1} \circ \widetilde{E}_n(\partial_n)$ where

$$\partial_n : X^n/X^{n-1} \xrightarrow{\simeq} C_i \to \Sigma X^{n-1} \to \Sigma X^{n-1}/X^{n-2}.$$

Note that $\widetilde{E}_n(\partial_n) = \partial$ is the connecting homomorphism in the long exact sequence

$$\ldots \to \widetilde{E}_n(X^n/X^{n-1}) \xrightarrow{\partial} \widetilde{E}_{n-1}(X^{n-1}) \to \widetilde{E}_{n-1}(X^n) \to \widetilde{E}_{n-1}(X^n/X^{n-1}) \to \ldots$$

Now, since h is natural and commutes with Σ, we have a commutative diagram:

$$\begin{tikzcd}
\pi_n^{ab}(X^n/X^{n-1}) \arrow[r, "\pi_n\partial_n"] \arrow[d, "h"] & \pi_n^{ab}(\Sigma X^{n-1}/X^{n-2}) \arrow[r, "\Sigma^{-1}"] \arrow[d, "h"] & \pi_{n-1}^{ab}(X^{n-1}/X^{n-2}) \arrow[d, "h"] \\
\widetilde{E}_n(X^n/X^{n-1}) \arrow[r, "\widetilde{E}_n(\partial_n)"] & \widetilde{E}_n(\Sigma X^{n-1}/X^{n-2}) \arrow[r, "\Sigma^{-1}"] & \widetilde{E}_{n-1}(X^{n-1}/X^{n-2}).
\end{tikzcd}$$

The top row is the differential of $C_n^{CW}(X)$. This shows that there is an isomorphism of chain complexes

$$C_*^E(X) \cong C_*^{CW}(X).$$

We construct a natural isomorphism

$$\alpha : \widetilde{E}_n(X) \to \ker(d_n)/\operatorname{im}(d_{n+1}) = \widetilde{H}_n(X)$$

using the diagram, which is based on the fact that we know the homology of a wedge of spheres. Here, we use the following fact.

Lemma 1.4. *$\widetilde{E}_n(Y) = 0$ if Y is a CW complex of dimension $\leq n$ and $n+1 \leq m$.*

Proof. If Y has dimension 0, then it is a set of points and this follows from the dimension axiom. If Y has dimension n, then consider the exact sequence:

$$\widetilde{E}_m(Y^{n-1}) \to \widetilde{E}_m(Y) \to \widetilde{E}_m(Y/Y^{n-1}) = \widetilde{E}_m(\bigvee D^n/\partial D^n)$$

The claim holds for Y^{n-1} by the induction hypothesis and for a wedge of spheres by the dimension axiom and the suspension isomorphism. Hence, it holds for Y. $\square$

This lemma implies that $\widetilde{E}_{n+1}(X^n/X^{n-1}) = 0$. We then have the following commutative diagram, where the columns are pieces of the long exact sequences for $X^k \subset X^{k+1}$.

$$\begin{tikzcd}
\widetilde{C}^{CW}_{n+1}(X) \arrow[r, equal] & \widetilde{E}_{n+1}(X^{n+1}/X^n) \arrow[d, "\partial"] \arrow[dr, "d_{n+1}"] & & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{E}_n(X^n) \arrow[r, "\pi^n_*"] \arrow[d, "i_*"] & \widetilde{E}_n(X^n/X^{n-1}) \arrow[r, "\partial"] \arrow[d, equal] \arrow[dr, "d_n"] & \widetilde{E}_{n-1}(X^{n-1}) \arrow[d, "\pi^{n-1}_*"] & \\
\widetilde{E}_n(X) \arrow[r, equal] & \widetilde{E}_n(X^{n+1}) \arrow[ur, dotted, "\alpha"'] \arrow[d] & \widetilde{C}^{CW}_n(X) & \widetilde{E}_{n-1}(X^{n-1}/X^{n-2}) \arrow[r, equal] & \widetilde{C}^{CW}_{n-1}(X) \\
& 0 & & &
\end{tikzcd}$$

Definition. For any y such that $i_*(y) = x$, let

$$\alpha(x) = \pi_*(y).$$

Well-defined. First, note that

$$d_n(\alpha(x)) = \pi^{n-1}_* \partial(\pi^n_*(y)) = 0$$

since $\partial \circ \pi^n_* = 0$. So $\alpha(x) \in \ker(d_n)$. Now, if $i_*(y') = x$, we have $i_*(y - y') = 0$, so that

$$y - y' = \partial(z).$$

Hence,

$$\pi^n_*(y - y') = \pi^n_* \partial(z) = d_{n+1}(z).$$

Hence, $y \equiv y' \mod \operatorname{im} d_{n+1}$. So, α is well-defined.

Injective. If $\alpha(x) = 0$, then $\pi^n_*(y) = 0$, but π^n_* is injective, so $y = 0$. Hence, $0 = i_*(y) = x$.

Surjective. Let $a \in \ker(d_n)$. Then, $a \in \ker \partial$ since π^{n-1}_* is injective. That means that $a = \pi^n_* y$. Then $\alpha(i_*(y)) = a$.

Naturality. All maps in the diagram are natural, hence so is α. Suspension is a natural isomorphism of chain complexes

$$\widetilde{C}^{CW}_n(X) = \pi^{ab}_n(X^n/X^{n+1}) \xrightarrow{\cong}{}^{\Sigma} \pi^{ab}_{n+1}(\Sigma X^n/X^{n+1}) \cong \widetilde{C}^{CW}_{n+1}(\Sigma X)$$

and everything in the above diagram commutes with the suspension isomorphism. Therefore, α commutes with Σ. $\square$