MATH 6280 - CLASS 27

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These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Reduced and relative homology and cohomology

Definition 1.1. For $A \subset X$ a sub-complex, let

$$C_n(A) \subset C_n(X)$$

as the cells of A are a subset of the cells of X.

- Let $C_*(X, A) = C_*(X)/C_*(A)$ and $H_*(X, A) = H_*(C_*(X, A))$.
- Let
$$C^*(X, A) = \ker(C^*(X) \to C^*(A)) = \mathrm{Hom}(C_*(X, A), \mathbb{Z})$$
and $H^*(X) = H^*(C^*(X, A))$.

We can give similar definitions for

$$H_*(X, A; M)$$

and

$$H^*(X, A; M).$$

Definition 1.2. For X a based CW-complex with based point $*$ a zero cell,

- Let $\widetilde{C}_*(X) = C_*(X)/C_*(*)$ and $\widetilde{H}_*(X) = H_*(\widetilde{C}_*(X))$.
- Let
$$\widetilde{C}^*(X) = \ker(C^*(X) \to C^*(*)) = \mathrm{Hom}(\widetilde{C}_*(X), \mathbb{Z})$$
and $\widetilde{H}^*(X) = H^*(\widetilde{C}^*(X))$.

Remark 1.3. Note that $C_*(X, A) = C_*(X)/C_*(A) \cong \widetilde{C}_*(X/A)$. Indeed, X/A has a CW-structure all cells in A identified to the base point and one cell for each cell not in A. Therefore, we have a natural isomorphism
$$H_*(X, A) \cong \widetilde{H}_*(X/A).$$
Similarly,
$$H^*(X, A) \cong \widetilde{H}^*(X/A)$$

Unreduced cohomology can be though of as a functor from the homotopy category of pairs of topological spaces to abelian groups:
$$H_*(-, -; M) : h\mathrm{CWpairs} \to \mathrm{Ab}$$
where
$$H_*(X; M) = H_*(X, \emptyset; M).$$

2. Eilenberg-Steenrod Axioms

2.1. Axioms for unreduced homology.

Definition 2.1. A family of functors
$$E_q(-) : h\mathrm{CWpairs} \to \mathrm{Ab}$$
indexed over the integers and natural transformations
$$\partial : E_q(X, A) \to E_{q-1}(A)$$
which satisfies the following axioms is called an unreduced homology theory.

(1) (Exactness) There is a long exact sequence
$$\ldots \to E_q(A) \to E_q(X) \to E_q(X, A) \to E_{q-1}(A) \to \ldots$$
(2) (Excision) If $(X; A, B)$ is a CW triad, then
$$E_q(A, A \cap B) \to E_q(X, B)$$

(3) (Additivity) If $(X, A) = \coprod_i (X_i, A_i)$, then

$$\bigoplus_i E_q(X_i, A_i) \xrightarrow{\cong} E_q(X, A)$$

Let M be an abelian group. The following additional property, which may or may not be satisfied, is called the *dimension axiom*:

- (Dimension for M) $E_*(*) = M$ concentrated in degree zero.

For (X, A) a pair of spaces, let $(\Gamma X, \Gamma A)$ be a CW-approximation. Defining

$$E_*(X, A) := E_*(\Gamma X, \Gamma A)$$

we have that the above axioms are equivalent to the following.

Definition 2.2. A family of functors

$$E_q(-;) : h\text{Top-pairs} \to \text{Ab}$$

indexed over the integers and natural transformations

$$\partial : E_q(X, A) \to E_{q-1}(A)$$

which satisfies the following axioms is called an unreduced homology theory.

(1) (Exactness) There is a long exact sequence

$$\ldots \to E_q(A) \to E_q(X) \to E_q(X, A) \to E_{q-1}(A) \to \ldots$$

(2) (Excision) If $(X; A, B)$ is an excisive triad, then

$$E_q(A, A \cap B) \to E_q(X, B)$$

(3) (Additivity) If $(X, A) = \coprod_i (X_i, A_i)$, then

$$E_q(X, A; M) \cong \bigoplus_i E_q(X_i, A_i)$$

(4) (Weak equivalence) If $f : (X, A) \to (Y, B)$ is a weak equivalence, then $E_*(f)$ is an isomorphism.

Let M be an abelian group. The following additional property, which may or may not be satisfied, is called the *dimension axiom*:

- (Dimension) $E_*(*) = M$ concentrated in degree zero.

2.2. **Axioms for reduced homology.**

Definition 2.3. A family of functors

$$\widetilde{E}_q(-) : h\,\mathrm{CWTop}_* \to \mathrm{Ab}$$

indexed over the integers

(1) (Exactness) If A is a subcomplex of X

$$\widetilde{E}_q(A) \to \widetilde{E}_q(X) \to \widetilde{E}_q(X/A)$$

is exact.

(2) (Suspension) There are natural isomorphisms

$$\Sigma : \widetilde{E}_q(X) \to \widetilde{E}_{q+1}(\Sigma X)$$

(3) (Additivity) If $X = \bigvee_i X_i$, then

$$\bigoplus_i \widetilde{E}_q(X_i) \xrightarrow{\cong} \widetilde{E}_q(X).$$

Let M be an abelian group. The following additional property, which may or may not be satisfied, is called the *dimension axiom*:

- (Dimension for M) $\widetilde{E}_*(S^0) = M$ concentrated in degree zero.

Again, for well based space X, let $(\Gamma X, \Gamma *) \to (X, *)$ be a CW approximation with $\Gamma *$ a point. Then we can define

$$\widetilde{E}_q(X) := \widetilde{E}_q(\Gamma X).$$

The axioms translate to:

Definition 2.4. A family of functors

$$\widetilde{E}_q(-) : h\mathrm{Top}_* \to \mathrm{Ab}$$

indexed over the integers (where Top_* is the category of well-based spaces) which satisfies the following axioms is called a reduced homology theory.

(1) (Exactness) If $A \to X$ is a cofibration, then

$$\widetilde{E}_q(A) \to \widetilde{E}_q(X) \to \widetilde{E}_q(X/A)$$

is exact.

(2) (Suspension) There are natural isomorphisms

$$\Sigma : \widetilde{E}_q(X) \to \widetilde{E}_{q+1}(\Sigma X)$$

(3) (Additivity) If $X = \bigvee_i X_i$, then

$$\bigoplus_i \widetilde{E}_q(X_i) \xrightarrow{\cong} \widetilde{E}_q(X).$$

(4) (Weak equivalence) If $f : X \to Y$ is a weak equivalence, then $\widetilde{E}_*(f)$ is an isomorphism.

Let M be an abelian group. The following additional property, which may or may not be satisfied, is called the *dimension axiom*:

- (Dimension for M) $\widetilde{E}_*(S^0) = M$ concentrated in degree zero.

Remark 2.5. Note that Exactness implies the following:

- If $A \xrightarrow{f} X \to C_f$ is a cofiber sequence, then

  $$\widetilde{E}_q(A) \to \widetilde{E}_q(X) \to \widetilde{E}_q(C_f)$$

  is exact.

Indeed, consider the commuative diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & X \\ & M_f \arrow[ur, "r"'] & \end{tikzcd}$$

then $A \to M_f$ is a cofibration so there is a commutative diagram, where $\widetilde{E}_q(r)$ is an isomorphism and the top row is exact:

$$\begin{tikzcd} \widetilde{E}_q(A) \arrow[r] \arrow[d, equal] & \widetilde{E}_q(M_f) \arrow[r] \arrow[d, "\widetilde{E}_q(r)", "\cong"'] & \widetilde{E}_q(M_f/A) \arrow[d, equal] \\ \widetilde{E}_q(A) \arrow[r] & \widetilde{E}_q(X) \arrow[r] & \widetilde{E}_q(C_f) \end{tikzcd}$$

Proposition 2.6. *Given a reduced cohomology theory $\widetilde{E}$ and a cofiber sequence $A \to X \to C_f$, there is a long exact sequence*

$$\ldots \longrightarrow \widetilde{E}_{n+1}(C_f) \longrightarrow \widetilde{E}_n(A) \longrightarrow \widetilde{E}_n(X) \longrightarrow \widetilde{E}_n(C_f) \longrightarrow \widetilde{E}_{n-1}(A) \longrightarrow \ldots$$

2.3. **Axioms for cohomology.** All of the above axioms generalize to contravariant functors

$$E^q(-), \quad \widetilde{E}^q(-)$$

with all arrows reversed in the obvious way. However, the Additivity axiom has a different flavor:

- (Unreduced Additivity) If $(X, A) = \coprod_i (X_i, A_i)$, then
$$E_q(X, A) \xrightarrow{\cong} \prod_i E^q(X_i, A_i).$$
- (Reduced Additivity) If $X = \bigvee_i X_i$, then
$$\widetilde{E}^q(X) \xrightarrow{\cong} \prod_i \widetilde{E}^q(X_i).$$

Remark 2.7 (Passage from reduced to unreduced)**.** Suppose that we are given an unreduced homology theory E. Let X pointed CW-complex. Note that

$$* \to X \to *$$

is the identity. This implies that the long exact sequence

$$\ldots \to E_{q+1}(X, *) \to E_q(*) \to E_q(X) \to E_q(X, *) \to E_{q-1}(*) \to \ldots$$

splits into split short exact sequences

$$0 \to E_q(*) \to E_q(X) \to E_q(X, *) \to 0$$

so that
$$E_*(X) \cong E_*(X, *) \oplus E_*(*).$$
Then, the functor
$$\widetilde{E}_*(X) = E_*(X, *)$$
gives a reduced homology theory.

Conversely, given a reduced homology theory $\widetilde{E}_*$ and a CW-pair (X, A). Then letting

$$E_*(X, A) = \widetilde{E}_*(X_+/A_+).$$

In particular,
$$E_*(X) = E_*(X, \emptyset) = \widetilde{E}_*(X_+/\emptyset_+) = \widetilde{E}_*(X_+).$$
This will give an unreduced homology theory.

Note that nothing in this remark appealed to the dimension axiom.