MATH 6280 - CLASS 26

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Cellular chains and cochains

Last time, we described the boundary in the cellular chain complex

$$C_n(X) = \mathbb{Z}\{I_n\} \cong \widetilde{H}'_{n-1}\left(\bigvee_{I_n} S_i^{n-1}\right)$$

as the map

$$\widetilde{H}'_{n-1}\left(\bigvee_{j\in I_n} S_j^{n-1}\right) \to \widetilde{H}'_{n-1}\left(\bigvee_{i\in I_{n-1}} S_i^{n-1}\right) \xrightarrow{\Sigma^{-1}} \widetilde{H}'_{n-2}\left(\bigvee_{i\in I_{n-1}} S_i^{n-2}\right)$$

where the first map is the one on $\widetilde{H}'_{n-1}$ induced by the composite

$$\bigvee_{j\in I_n} S_j^{n-1} \xrightarrow{\Phi_n} X^{n-1} \to X^{n-1}/X^{n-2} = \bigvee_{i\in I_{n-1}} D_i^{n-1}/S_i^{n-2} \cong \bigvee_{i\in I_{n-1}} S_i^{n-1}.$$

Remark 1.1. Note that

$$C_n(X) = \mathbb{Z}\{I_n\} \cong \widetilde{H}'_{n-1}\left(\bigvee_{I_n} S_i^{n-1}\right) \cong^{\Sigma} \widetilde{H}'_n\left(\bigvee_{I_n} S^n\right) = \widetilde{H}'_n\left(X^n/X^{n-1}\right).$$

Let $i : X^{n-1} \to X^n$ be the inclusion and let the quotient $\psi : X^n \cup CX^{n-1} \to X^n/X^{n-1}$ have homotopy inverse ψ^{-1}. Let ∂_n be the composite

$$X^n/X^{n-1} \xrightarrow{\psi^{-1}} X^n \cup CX^{n-1} \to \Sigma X^{n-1} \to \Sigma(X^{n-1}/X^{n-2}).$$

Then, up to sign, the differential $d_n : C_n(X) \to C_{n-1}(X)$ can be identified with

$$\widetilde{H}'_n(X^n/X^{n-1}) \xrightarrow{\widetilde{H}'_n(\partial_n)} \widetilde{H}'_n(\Sigma(X^{n-1}/X^{n-2})) \xrightarrow{\Sigma^{-1}} \widetilde{H}'_{n-1}(X^{n-1}/X^{n-2}).$$

To show this, note that the top row of the following diagram is a homotopy cofiber sequence:

$$\begin{tikzcd}
\bigvee_{I_n} S^{n-1} \arrow[r, "\Phi"] & X^{n-1} \arrow[r, "i"] & X^n \arrow[r] & X^n \cup CX^{n-1} \arrow[r] \arrow[d] & (X^n \cup CX^{n-1}) \cup CX^n \arrow[d] & \\
 & & \Sigma\left(\bigvee_{I_n} S^{n-1}\right) \arrow[r, equal] & X^n/X^{n-1} \arrow[r, "-\Sigma\Phi"] & \Sigma X^{n-1} \arrow[r] & \Sigma X^{n-1}/X^{n-2} \arrow[d] \\
 & & & & & \Sigma\left(\bigvee_{I_{n-2}} S^{n-1}\right)
\end{tikzcd}$$

We have already proved that the square commutes up to homotopy. So, up to a sign, $\widetilde{H}'_n(\partial_n)$ is the suspension of the map

$$\bigvee_{I_n} S^{n-1} \to X^{n-1} \to X^{n-1}/X^{n-2} \cong \bigvee_{I_{n-2}} S^{n-1}.$$

(You can make that sign go away by being clever about your various homeomorphisms (see *Concise*).)

Now, we can see that $d_n \circ d_{n-1} = 0$. Indeed, we have $d_n = \Sigma^{-1} \circ \widetilde{H}'_n(\partial_n)$ for

$$\partial_n : X^n/X^{n-1} \xrightarrow{\psi^{-1}} C_{i_{n-1}} \to \Sigma X^{n-1} \to \Sigma(X^{n-1}/X^{n-2})$$

and $d_{n-1} = \Sigma^{-1} \circ \widetilde{H}'_{n-1}(\partial_{n-1})$ for

$$\partial_{n-1} : X^{n-1}/X^{n-2} \xrightarrow{\psi^{-1}} C_{i_{n-2}} \to \Sigma X^{n-2} \to \Sigma(X^{n-2}/X^{n-3}).$$

So, it's enough to check that

$$\Sigma(\partial_{n-1}) \circ \partial_n$$

is null homotopic, since the following diagram commutes

$$\begin{tikzcd}
\widetilde{H}'_n(X^n/X^{n-1}) \arrow[r, "\widetilde{H}'_n(\partial_n)"] \arrow[dr, "d_n"'] & \widetilde{H}'_n(\Sigma X^{n-1}/X^{n-2}) \arrow[r, "\widetilde{H}'_n(\Sigma\partial_{n-1})"] \arrow[d, "\Sigma^{-1}"'] \arrow[dr, "d_{n-1}"] & \widetilde{H}'_n(\Sigma^2 X^{n-2}/X^{n-3}) \arrow[d, "\Sigma^{-1}"] \\
 & \widetilde{H}'_n(X^{n-1}/X^{n-2}) \arrow[r, "\widetilde{H}'_n(\partial_{n-1})"] \arrow[dr, "d_{n-1}"'] & \widetilde{H}'_n(\Sigma X^{n-2}/X^{n-3}) \arrow[d, "\Sigma^{-1}"] \\
 & & \widetilde{H}'_n(X^{n-2}/X^{n-3})
\end{tikzcd}$$

You can check this with the commutative diagram below, where the composite of the top row is null-homotopic:

$$\begin{tikzcd}
X^n \cup CX^{n-1} \arrow[r] \arrow[d, "\simeq"'] & \Sigma X^{n-1} \arrow[r] \arrow[d] & \Sigma(X^{n-1} \cup CX^{n-2}) \arrow[r] \arrow[d] & \Sigma^2 X^{n-2} \arrow[d] \\
X^n/X^{n-1} \arrow[r, "\partial_n"] & \Sigma X^{n-1}/X^{n-2} \arrow[r, "="] & \Sigma X^{n-1}/X^{n-2} \arrow[r, "\Sigma\partial_{n-1}"] & \Sigma^2 X^{n-2}/X^{n-3}
\end{tikzcd}$$

Definition 1.2. Let X be a CW–complex. Let M be an abelian group.

(1) The *cellular chain complex with coefficients* M is

$$C_*(X;M) = C_*(X) \otimes_{\mathbb{Z}} M.$$

(2) The *cellular cochain complex* of X is

$$\mathrm{Hom}_{\mathbb{Z}}(C_*(X), \mathbb{Z})$$

with differential $d^{n+1} = \mathrm{Hom}_{\mathbb{Z}}(d_n, \mathbb{Z}) : C^n \to C^{n+1}$.

(3) The *cellular cochain complex with coefficients* M is

$$C^*(X;M) = \mathrm{Hom}_{\mathbb{Z}}(C_*(X), M).$$

If M is a commutative ring, then

$$C^*(X;M) = \mathrm{Hom}_{\mathbb{Z}}(C_*(X), M) = \mathrm{Hom}_M(C_*(X,M), M).$$

Recall that if X and Y are CW-complexes and X has n–cells I_n and Y has n–cells J_n, then $X \times Y$ has a CW-structure whose n

Proposition 1.3. *If X and Y are CW-complexes, and $X \times Y$ is given the CW-structure with cells the product of the cells in X and Y. . There is an isomorphism of chain complexes*

$$C_*(X) \otimes C_*(Y) \cong C_*(X \times Y).$$

Let $i \in I_p(X)$ and $j \in J_q(Y)$. It is given by the map

$$\phi : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

which, up to sign, sends $\phi([i] \otimes [j])$ to $[i \times j]$.

2. Homology and Cohomology

Definition 2.1. Let X be a CW–complex. Let M be an abelian group.

(1) The *cellular homology of X*, is the homology of the cellular chain complex of X:

$$H_*(X) = H_*(C_*(X))$$

(2) The *cellular homology of X with coefficients in M* is

$$H_*(X; M) = H_*(C_*(X; M))$$

The *cellular cohomology of X*, is the homology of the cellular chain complex of X:

$$H^*(X) = H^*(C^*(X))$$

(3) The *cellular cohomology of X with coefficients in M* is

$$H^*(X; M) = H^*(C^*(X; M))$$

Remark 2.2.
- If $X \xrightarrow{f} Y$ is a cellular map, then it induces a map $C_*(X) \to C_*(Y)$, which in turn gives a map $H_*(X) \to H_*(Y)$. In cohomology, this gives a map $H^*(Y) \to H^*(X)$.
- Let I have the CW-structure with two 0–cells $[0]$ and $[1]$ and one 1–cell attached in the obvious way. Then, a cellular homotopy between cellular maps $g, f : X \to Y$, namely $h : X \times I \to Y$ gives a map $C_*(X) \otimes C_*(I) \to C_*(Y)$. This data is equivalence to a chain homotopy between $C_*(f)$ and $C_*(g)$. Some, homotopic maps induce isomorphism on homology and cohomology.
- If X is homotopy equivalent to Y, then $H_*(X) \cong H_*(Y)$. Since X and Y are CW–complexes, this holds for weak equivalences also.

3. Extending the definitions to all spaces

Definition 3.1. Let X be any space and choose $\Gamma X \to X$ a CW-approximation. Then

$$H_*(X) := H_*(\Gamma X)$$

and

$$H^*(X) := H^*(\Gamma X).$$