MATH 6280 - CLASS 25

ContentsThese notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

Recall from last time that:

Lemma 0.1. *If X is a wedge of n–spheres, then $\pi_n X$ if $n \geq 2$ (and $\pi_1(X)^{ab}$) is the free abelian group generated by the inclusions of the summands.*

Definition 0.2. Let X be $n-1$–connected. Define

$$\widetilde{H}'_n(X) = \begin{cases} \mathbb{Z}\{\pi_0 X - \{*\}\} & n = 0 \\ \pi_1 X / [\pi_1 X, \pi_1 X] & n = 1 \\ \pi_n X & n > 1. \end{cases}$$

Lemma 0.3. *If X is a wedge of n-spheres, then ΣX is a wedge of $n+1$–spheres and the map which sends $f : S^n \to X$ to $\Sigma f : S^{n+1} \to \Sigma X$ induces an isomorphism*

$$\Sigma : \widetilde{H}'_n(X) \to \widetilde{H}'_{n+1}(\Sigma X)$$

is an isomorphism.

Proof. This follows from our description of $\widetilde{H}'_n(X)$ as the free abelian on the inclusions of the summands. □

1. Cellular chains and cochains

Before proving this, let's have a quick review of cellular homology.

Definition 1.1. Let X be a CW–complex. The *cellular chain complex* of X. For $n \geq 0$, let $C_n(X)$ be the free abelian group generated by the n-cells of X. That is, if

$$\begin{tikzcd} \cup_{i\in I_n} S_i^{n-1} \arrow[r, "\Phi_n=\cup\phi_i^n"] \arrow[d] & X^{n-1} \arrow[d] \\ \cup_{i\in I_n} D_i^n \arrow[r] & X^n \end{tikzcd}$$

then $C_n(X) = \mathbb{Z}\{I_n\}$. Note that

$$C_n(X) = \mathbb{Z}\{I_n\} \cong \widetilde{H}'_{n-1}\left(\bigvee_{I_n} S_i^{n-1}\right)$$

Note that there is a map

$$\bigvee_{j\in I_n} S_j^{n-1} \xrightarrow{\Phi_n} X^{n-1} \to X^{n-1}/X^{n-2} = \bigvee_{i\in I_{n-1}} D_i^{n-1}/S_i^{n-2} \cong \bigvee_{i\in I_{n-1}} S_i^{n-1}.$$

Then the differential $d_n : C_n(X) \to C_{n-1}(X)$ is the induced map on $\widetilde{H}'_{n-1}(X)$.

More concretely, for $j \in I_n$ and $\phi_j^n : S_j^{n-1} \to X^{n-1}$, the composite

$$S_j^{n-1} \xrightarrow{\phi_j^n} X^{n-1} \to X^{n-1}/X^{n-2} = \bigvee_{i\in I_{n-1}} S_i^{n-1} \xrightarrow{\pi_i} S_i^{n-1}$$

is an element of $\pi_{n-1}S^{n-1}$ and we let $a_{i,j} \in \mathbb{Z}$ be its degree. Then the differential

$$d_n : C_n(X) \to C_{n-1}(X)$$

is given by

$$d_n([j]) = \sum_{i\in I_{n-1}} a_{i,j}[i].$$

If $n = 1$, then $d_1([j]) = [\phi_j^1(1)] - [\phi_j^1(-1)]$.

Example 1.2. Let $\mathbb{R}P^3$ have the usual cell structure with one cell in each dimension $0 \leq d \leq 3$. Then

$$C_n(\mathbb{R}P^3) = \begin{cases} \mathbb{Z}\{i_n\} & 0 \leq n \leq 3 \\ 0 & n > 3 \end{cases}$$

The attaching maps are the double covers, so we need to compute the degrees of

$$\phi^0 : S^0 \to *$$
$$\phi^1 : S^1 \to \mathbb{R}P^1 \to \mathbb{R}P^1/* \simeq S^1$$
$$\phi^2 : S^2 \to \mathbb{R}P^2 \to \mathbb{R}P^2/\mathbb{R}P^1 \simeq S^2.$$

We have that

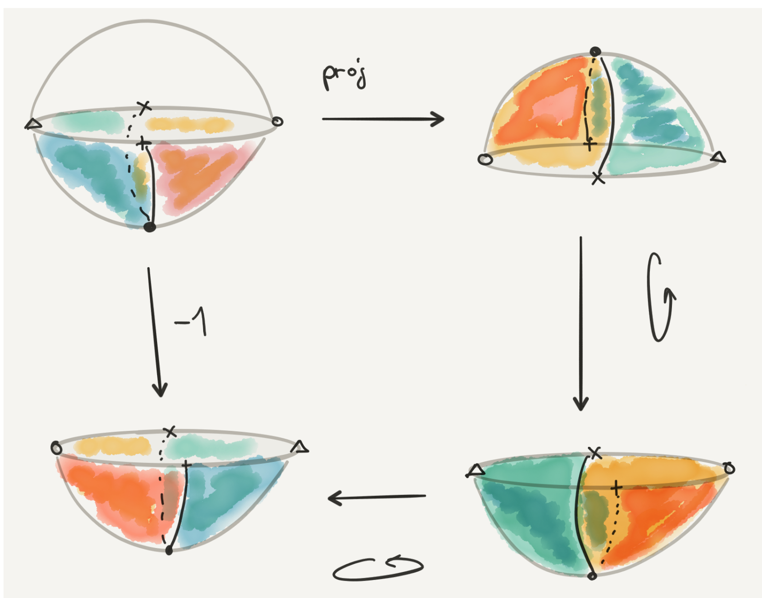
$$d_1(i_1) = i_0 - i_0 = 0$$

and $\phi^1 : S^1 \to S^1$ is the multiplication by 2 map so that

$$d_2(i_2) = 2i_1$$

The next picture explains how to see that $\phi^2 : S^2 \to S^2$ is homotopic to $(\mathrm{id} \vee -\mathrm{id}) \circ \nabla \simeq *$ so that

$$d_3(i_3) = 0.$$



Remark 1.3. Note that

$$C_n(X) = \mathbb{Z}\{I_n\} \cong \widetilde{H}'_{n-1}\left(\bigvee_{I_n} S_i^{n-1}\right) \cong \widetilde{H}'_n\left(\bigvee_{I_n} D_i^n/S_i^{n-1}\right) = \widetilde{H}'_n\left(X^n/X^{n-1}\right).$$

Let $i : X^{n-1} \to X^n$ be the inclusion and let the quotient $\psi : C_i \to X^n/X^{n-1}$ have homotopy inverse ψ^{-1}. Let ∂_n be the composite

$$X^n/X^{n-1} \xrightarrow{\psi^{-1}} C_i \to \Sigma X^{n-1} \to \Sigma(X^{n-1}/X^{n-2}).$$

We will see next time how, up to sign, the differential $d_n : C_n(X) \to C_{n-1}(X)$ can be identified with

$$\widetilde{H}'_n X^n/X^{n-1} \xrightarrow{\widetilde{H}'_n(\partial_n)} \widetilde{H}'_n(\Sigma(X^{n-1}/X^{n-2})) \xrightarrow{\Sigma^{-1}} \widetilde{H}'_{n-1}(X^{n-1}/X^{n-2}).$$