MATH 6280 - CLASS 22

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Triads and homotopy excision - continued

Recall:

Theorem 1.1 (Homotopy excision)**.** *Let $(X; A, B)$ be an excisive triad with $A \cap B \neq \emptyset$. Suppose that*

- *$(A, A \cap B)$ is n–connected for $n \geq 1$.*
- *$(B, A \cap B)$ is m–connected for $m \geq 0$.*

Then $(A, A \cap B) \to (X, B)$ is an $m + n$ equivalence.

We proved:

Proposition 1.2. *Suppose that $f : X \to Y$ is an n–equivalence and X is m–connected for $n \geq 1$. Then $(M_f, X) \to (C_f, *)$ is a $n + m + 1$-equivalence.*

As an immediate consequence, we have:

Corollary 1.3. *Suppose that $f : X \to Y$ is an n–equivalence and X is m–connected for $n \geq 1$, then $(C_f, *)$ is n-connected.*

Corollary 1.4. *Let $n \geq 0$. Let $i : A \to X$ be a cofibration which is an $n+1$–equivalence. Then $(X, A) \to (X/A, *)$ is a $2n+2$-equivalence if A is n–connected.*

Proof. Use the commutative diagram

$$\begin{array}{ccc} (M_i, A) & \longrightarrow & (C_i, *) \\ \downarrow & & \downarrow \\ (X, A) & \longrightarrow & (X/A, *) \end{array}$$

Because i is a cofibration, the vertical arrows are homotopy equivalences of pairs. The connectivity of the two arrow follows from the previous proposition. □

Theorem 1.5 (Freudenthal Suspension)**.** *Let X be an n–connected based space with a non-degenerated base point. Then the map*

$$\Sigma : \pi_q X \to \pi_{q+1} \Sigma X$$

which sends f to Σf is an isomorphism for $q \leq 2n$ and surjective for $q = 2n+1$.

Proof. Let $CX = X \times I/(* \times I \cup \{0\} \times X)$ be the reduced cone. Using the long exact sequence on homotopy groups and the fact that CX is contractible, we have that

$$\pi_{q+1}(CX, X, *) \xrightarrow{\partial} \pi_q X$$

is an isomorphism for all q. Further, given a map $(I^q, \partial I^q) \xrightarrow{f} (X, *)$, a lift is given by the map

$$f \times I : (I^q \times I, \partial(I^q \times I), J^{q+1}) \to (CX, X, *).$$

Note that composing $f \times I$ with the quotient map $\pi : (CX, X) \to (CX/X, *)$ gives the map Σf. That is, the following diagram commutes:

$$\begin{tikzcd} \pi_{q+1}(CX, X, *) \arrow[r, "\pi"] \arrow[d, "\partial"'] & \pi_{q+1}(CX/X, *) \cong \pi_{q+1}\Sigma X \\ \pi_q(X) \arrow[ur, "\Sigma"'] & \end{tikzcd}$$

where π is induced by the quotient map. Note that X is n–connected and CX is contractible. So $X \to CX$ is an $n+1$–equivalence, so $(CX, X) \to (CX/X, *)$ is a $2n+2$-equivalence. We have that π is a surjection if $q+1 = 2n+2$ and an isomorphism if $q+1 < 2n+2$. □

Exercise 1.6. Prove that if X is n–connected, then ΣX is $n+1$–connected.

Corollary 1.7. *The suspension $\Sigma : \pi_n S^n \to \pi_{n+1} S^{n+1}$ is an isomorphism for all $n \geq 1$. In particular, $\pi_n S^n \cong \mathbb{Z}$ for all $n \geq 1$.*

Proof. We know that $\pi_1 S^1 \cong \mathbb{Z}$. Since S^n is $n-1$–connected, so that Σ is an isomorphism when $q \leq 2n-2$ and a surjection if $q = 2n-1$. If $n \geq 2$, then $n \leq 2n-2$ so $\pi_n S^n \to \pi_{n+1} S^{n+1}$ is an isomorphism. If $n = 1$, $\pi_1 S^1 \to \pi_2 S^2$ is surjective. Consider the Hopf fibration

$$S^1 \to S^3 \to S^2.$$

We have $0 = \pi_2 S^3 \to \pi_2 S^2 \to \pi_1 S^1 \to \pi_1 S^3 = 0$ is exact, so $\pi_2 S^2 \cong \mathbb{Z}$. Hence, $\pi_n S^n \cong \mathbb{Z}$ for all $n \geq 1$. Finally, $\Sigma : \pi_1 S^1 \to \pi_2 S^2$ is surjective and there are no surjective group homomorphism $\mathbb{Z} \to \mathbb{Z}$ which is not an isomorphism, Σ is an isormorphism. □

Definition 1.8. The $id : S^n \to S^n$ gives a canonical isomorphism $\deg : \pi_n S^n \to \mathbb{Z}$, which we call the *degree*. The degree of a map $f : S^n \to S^n$ is just $\deg(f)$.

Definition 1.9. The stable homotopy groups of a based space X are

$$\pi_q^s X = \operatorname{colim}_n \pi_{q+n} \Sigma^n X$$

Corollary 1.10. $\pi_0^s S^0 = \mathbb{Z}$

2. Some remarks about homotopy fiber and cofiber

Let $f : X \to Y$ be a map of based spaces. There is always a map

$$\eta : P_f \to \Omega C_f$$

given by

$$\eta(x, \alpha) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ x \wedge (2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

That is, $\eta(x, \alpha)$ first does the path α in Y and then connects to the based point via the the cone coordinate.

It's adjoint is $\epsilon : \Sigma P_f \to C_f$, with, as usual $\epsilon(x, \alpha, t) = \eta(x, \alpha)(t)$. If you think of P_f as some sort of loops and C_f as some sort of suspension, this is the analogue of the units of the $\Sigma - \Omega$–adjunction

$$\eta : X \to \Omega \Sigma X \qquad\qquad \epsilon : \Sigma \Omega X \to X$$

Exercise 2.1 (See *Concise*, Chapter 10, Section 7)**.** Identify maps that will make the following diagram commutative up to homotopy:

$$\begin{tikzcd}
 & \Sigma\Omega P_f \arrow[r] \arrow[d] & \Sigma\Omega X \arrow[r] \arrow[d] & \Sigma\Omega Y \arrow[r] \arrow[d] & \Sigma P_f \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
\Omega Y \arrow[r] \arrow[d, equal] & P_f \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C_f \arrow[r] \arrow[d] & \Sigma X \\
\Omega Y \arrow[r] & \Omega C_f \arrow[r] & \Omega\Sigma X \arrow[r] & \Omega\Sigma Y \arrow[r] & \Omega\Sigma C_f &
\end{tikzcd}$$

Next time, we will prove:

Proposition 2.2. *If $f : X \to Y$ is a map between n–connected spaces for $n \geq 1$, then*

$$\eta : F_f \to \Omega C_f$$

is a $2n$-equivalence.