## MATH 6280 - CLASS 22

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These notes are based on

- Algebraic Topology from a Homotopical Viewpoint, M. Aguilar, S. Gitler, C. Prieto
- A Concise Course in Algebraic Topology, J. Peter May
- More Concise Algebraic Topology, J. Peter May and Kate Ponto
- Algebraic Topology, A. Hatcher

1. TRIADS AND HOMOTOPY EXCISION - CONTINUED

Recall:

**Theorem 1.1** (Homotopy excision). Let (X; A, B) be an excisive triad with  $A \cap B \neq \emptyset$ . Suppose that

- $(A, A \cap B)$  is n-connected for  $n \ge 1$ .
- $(B, A \cap B)$  is m-connected for  $m \ge 0$ .

Then  $(A, A \cap B) \to (X, B)$  is an m + n equivalence.

We proved:

**Proposition 1.2.** Suppose that  $f: X \to Y$  is an *n*-equivalence and X is *m*-connected for  $n \ge 1$ . Then  $(M_f, X) \to (C_f, *)$  is a n + m + 1-equivalence.

As an immediate consequence, we have:

**Corollary 1.3.** Suppose that  $f : X \to Y$  is an *n*-equivalence and X is *m*-connected for  $n \ge 1$ , then  $(C_f, *)$  is *n*-connected.

**Corollary 1.4.** Let  $n \ge 0$ . Let  $i : A \to X$  be a cofibration which is an n + 1-equivalence. Then  $(X, A) \to (X/A, *)$  is a 2n + 2-equivalence if A is n-connected.

Proof. Use the commutative diagram

Because i is a cofibration, the vertical arrows are homotopy equivalences of pairs. The connectivity of the two arrow follows from the previous proposition.

**Theorem 1.5** (Freudenthal Suspension). Let X be an n-connected based space with a non-degenerated base point. Then the map

$$\Sigma: \pi_q X \to \pi_{q+1} \Sigma X$$

which sends f to  $\Sigma f$  is an isomorphism for  $q \leq 2n$  and surjective for q = 2n + 1.

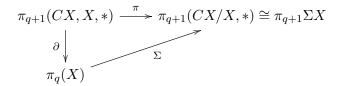
*Proof.* Let  $CX = X \times I/(* \times I \cup \{0\} \times X)$  be the reduced cone. Using the long exact sequence on homotopy groups and the fact that CX is contractible, we have that

$$\pi_{q+1}(CX, X, *) \xrightarrow{\partial} \pi_q X$$

is an isomorphism for all q. Further, given a map  $(I^q, \partial I^q) \xrightarrow{f} (X, *)$ , a lift is given by the map

$$f \times I : (I^q \times I, \partial(I^q \times I), J^{q+1}) \to (CX, X, *).$$

Note that composing  $f \times I$  with the quotient map  $\pi : (CX, X) \to (CX/X, *)$  gives the map  $\Sigma f$ . That is, the following diagram commutes:



where  $\pi$  is induced by the quotient map. Note that X is n-connected and CX is contractible. So  $X \to CX$  is an n + 1-equivalence, so  $(CX, X) \to (CX/X, *)$  is a 2n + 2-equivalence. We have that  $\pi$  is a surjection if q + 1 = 2n + 2 and an isomorphism if q + 1 < 2n + 2.

**Exercise 1.6.** Prove that if X is *n*-connected, then  $\Sigma X$  is n + 1-connected.

**Corollary 1.7.** The suspension  $\Sigma : \pi_n S^n \to \pi_{n+1} S^{n+1}$  is an isomorphism for all  $n \ge 1$ . In particular,  $\pi_n S^n \cong \mathbb{Z}$  for all  $n \ge 1$ .

Proof. We know that  $\pi_1 S^1 \cong \mathbb{Z}$ . Since  $S^n$  is n-1-connected, so that  $\Sigma$  is an isomorphism when  $q \leq 2n-2$  and a surjection if q = 2n-1. If  $n \geq 2$ , then  $n \leq 2n-2$  so  $\pi_n S^n \to \pi_{n+1} S^{n+1}$  is an isomorphism. If n = 1,  $\pi_1 S^1 \to \pi_2 S^2$  is surjective. Consider the Hopf fibration

$$S^1 \to S^3 \to S^2$$
.

We have  $0 = \pi_2 S^3 \to \pi_2 S^2 \to \pi_1 S^1 \to \pi_1 S^3 = 0$  is exact, so  $\pi_2 S^2 \cong \mathbb{Z}$ . Hence,  $\pi_n S^n \cong \mathbb{Z}$  for all  $n \ge 1$ . Finally,  $\Sigma : \pi_1 S^1 \to \pi_2 S^2$  is surjective and there are no surjective group homomorphism  $\mathbb{Z} \to \mathbb{Z}$  which is not an isomorphism,  $\Sigma$  is an isomorphism.  $\square$ 

**Definition 1.8.** The  $id: S^n \to S^n$  gives a canonical isomorphism deg :  $\pi_n S^n \to \mathbb{Z}$ , which we call the *degree*. The degree of a map  $f: S^n \to S^n$  is just deg(f).

**Definition 1.9.** The stable homotopy groups of a based space X are

$$\pi_q^s X = \operatorname{colim}_n \pi_{q+n} \Sigma^n X$$

Corollary 1.10.  $\pi_0^s S^0 = \mathbb{Z}$ 

## 2. Some remarks about homotopy fiber and cofiber

Let  $f:X\to Y$  be a map of based spaces. There is always a map

$$\eta: P_f \to \Omega C_f$$

given by

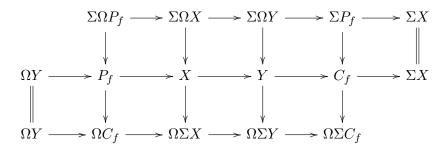
$$\eta(x,\alpha) = \begin{cases} \alpha(2t) & 0 \le t \le 1/2\\ x \land (2t-1) & 1/2 \le t \le 1. \end{cases}$$

That is,  $\eta(x, \alpha)$  first does the path  $\alpha$  in Y and then connects to the based point via the the cone coordinate.

It's adjoint is  $\epsilon : \Sigma P_f \to C_f$ , with, as usual  $\epsilon(x, \alpha, t) = \eta(x, \alpha)(t)$ . If you think of  $P_f$  as some sort of loops and  $C_f$  as some sort of suspension, this is the analogue of the units of the  $\Sigma - \Omega$ -adjunction

$$\eta: X \to \Omega \Sigma X \qquad \qquad \epsilon: \Sigma \Omega X \to X$$

**Exercise 2.1** (See *Concise*, Chapter 10, Section 7). Identify maps that will make the following diagram commutative up to homotopy:



Next time, we will prove:

**Proposition 2.2.** If  $f: X \to Y$  is a map between *n*-connected spaces for  $n \ge 1$ , then

$$\eta: F_f \to \Omega C_f$$

is a 2n-equivalence.