MATH 6280 - CLASS 21

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Triads and homotopy excision

Definition 1.1. (1) An *excisive triad*, denoted $(X; A, B)$, consists of a space X and subspaces A and B of X such that $X = \text{int}(A) \cup \text{int}(B)$.
(2) A *CW-triad* denoted $(X; A, B)$, consists of a CW-complex X and subcomplexes A and B such that $X = A \cup B$.

Here are two results which we will not prove.

Theorem 1.2. *A map $e : (X; A, B) \to (X'; A', B')$ of excisive triads which induces weak equivalences $A \simeq A'$, $B \simeq B'$ and $A \cap B \simeq A' \cap B'$ induces a weak equivalence $X \simeq X'$.*

Corollary 1.3. *An excisive triad $(X; A, B)$, then there is a CW-triad $e : (\Gamma X; \Gamma A, \Gamma B) \to (X, A, B)$ with e inducing weak equivalences $\Gamma A \simeq A$, $\Gamma B \simeq B$, $\Gamma A \cap \Gamma B \to A \cap B$, and, $\Gamma X \simeq X$. The approximation can be made functorial up to homotopy. Further, if $(A, A \cap B)$ is n–connected, we can choose $(\Gamma A, \Gamma A \cap \Gamma B)$ to have no cells for $q \leq n$ and the same holds for $(B, A \cap B)$.*

Recall the following fact about homology:

Theorem 1.4 (Excision)**.** *Given excisive triad, denoted $(X; A, B)$, the inclusion $(B, A \cap B) \to (X, A)$ gives an isomorphism on relative homology*

$$H_*(B, A \cap B) \to H_*(X, A).$$

Here, the relative homology groups $H_*(X, A)$ are computed using the complex $C_*(X, A) = C_*(X)/C_*(A)$.

Often, we use excision when we want to compute $H_*(X, B)$ by choosing A to be a neighborhood of $X - B$. Then what this is saying is that relative homology is somehow local: you only need chains in A relative to $A \cap B$ to compute it.

Some important results one can deduce from excision are:

(1) Homology of quotients: If $A \to X$ is a cofibration, then $H_*(X, A) \cong \widetilde{H}(X/A)$.
(2) Suspension isomorphism: $\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$
(3) Local orientations: By excision, $H_n(X, X - \{x\}) \cong H_n(U, U - \{x\})$. If X is an n-manifold, then $U \cong \mathbb{R}^n$ and

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \widetilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \cong \widetilde{H}_{n-1} S^{n-1} \cong \mathbb{Z}.$$

A local orientation is a choice of generator of $\mathbb{Z}$.

In homotopy, we don't quite get as good of a result.

Theorem 1.5 (Homotopy excision)**.** *Let $(X; A, B)$ be an excisive triad with $A \cap B \neq \emptyset$. Suppose that*

- *$(A, A \cap B)$ is n–connected for $n \geq 1$.*
- *$(B, A \cap B)$ is m–connected for $m \geq 0$.*

Then $(A, A \cap B) \to (X, B)$ is an $m + n$ equivalence.

Definition 1.6. Recall,

(1) (X, A) is n–connected if $\pi_0 A \to \pi_0 X$ is surjective and $\pi_q(X, A) = 0$ for $1 \leq q \leq n$.
(2) $(X, A) \xrightarrow{f} (Y, B)$ is an n–equivalence if
   - $f(i_*(\pi_0(A))) = i_*(\pi_0(B))$, where i are the inclusions $A \to X$ and $B \to Y$.
   - $\pi_q f$ is is an isomomorphism for $1 \leq q < n$ and surjective if $q = n$.

Let's see some consequences of excision.

First, we need the following lemma:

Lemma 1.7. *If $f : X \to Y$ is an n–equivalence, then (M_f, X) is n-connected.*

Proof. We use the long exact sequence on homotopy groups for the pair.

$$\pi_n X \to \pi_n M_f \to \pi_n(M_f, X) \to \pi_{n-1} X \to \pi_{n-1} M_f \to \dots$$

Since $X \to Y$ is an n equivalence and factors through $X \to M_f \xrightarrow{\simeq} Y$, then $X \to M_f$ is an n–equivalence. The claim follows. □

Proposition 1.8. *Suppose that $f : X \to Y$ is an n–equivalence and X is m–connected for $n \geq 1$. Then $(M_f, X) \to (C_f, *)$ is a $n + m + 1$-equivalence.*

Proof. We use excision. Consider the following subsets of C_f:

$$A = Y \cup_f X \times [0, 2/3]/(X \times 1) \qquad B = X \times [1/3, 1]/(X \times 1).$$

Note that A and B intersect in a collar $A \cap B = X \times [1/3, 2/3]$. We have homotopy equivalences

$$(M_f, X) \simeq (A, A \cap B)$$
$$(C_f, *) \simeq (C_f, B),$$

and up to homotopy, $(M_f, X) \to (C_f, *)$ factors through $(A, A \cap B) \to (C_f, B)$. So we will use the excisive triad

$$(C_f; A, B)$$

We need to prove that

- $(A, A \cap B)$ is n–connected
- $(B, A \cap B)$ is $m + 1$–connected.

The first is obvious since $(M_f, X) \simeq (A, A \cap B)$. For the second, note that there is a homotopy equivalence

$$(B, A \cap B) \simeq (CX, X)$$

and $\pi_{q+1}(CX, X) \cong \pi_q X$ by the long exact sequence on homotopy groups for a pair. So the connectivity of X gives the claim. □