MATH 6280 - CLASS 20

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Cellular Approximation

Last time, we proved:

Theorem 1.1. *Let $f : (X, A) \to (Y, B)$ be any map between CW–complexes. Then f is homotopic relative to A to a cellular map. That is, there is a homotopy $H : X \times I \to Y$ which is constant on A such that $H(x, 0) = f(x)$ and $H(-, 1)|_{X^n} \subset Y^n$.*

The following is now a direct consequence of the previous result:

Exercise 1.2. Any two maps $f : X \to Y$ between CW–complexes is homotopic to a cellular map. Any homotopic cellular maps are homotopic via a cellular homotopy, that is, a homotopy $H : X \times I \to Y$ which is a cellular map.

2. Approx of spaces by CW-complexes

Let's warm up.

Lemma 2.1. *Let X be any space. Fix $n \geq 0$. There exists a space Y and a map $i : X \to Y$ such that i induces isomorphisms*

$$\pi_q Y \cong \begin{cases} \pi_q X & 0 \leq q \leq n \\ 0 & q = n+1. \end{cases}$$

Proof. To do this, let J be a set of representatives each element $[j] \in \pi_{n+1}X$. Consider the pushout:

$$\begin{tikzcd} \sqcup_J S^{n+1} \arrow[r, "\sqcup j"] \arrow[d] & X \arrow[d] \\ \sqcup_J D^{n+2} \arrow[r] & Y. \end{tikzcd}$$

Note that (Y, X) is a relative CW complex and that $Y^{n+1} = X$, so that $X \to Y$ is an $n+1$ equivalence. Therefore, it is an isomorphism on π_q for $0 \leq q \leq n$ and surjective on π_{n+1}.

Choose the standard CW structure on S^{n+1}. Given any map $f : (S^{n+1}, *) \to (Y, X)$ is homotopic (relative to the base point), to a cellular map, i.e., one that factors through $Y^{n+1} = X$. It follows that $f \simeq j$ for some $j \in J$. However, $j : S^{n+1} \to X \to Y$ extends over the disk by construction, so is null homotopic. Hence, $\pi_{n+1}Y = 0$. ☐

Exercise 2.2. Construct spaces Y_n and maps $X \xrightarrow{i} Y_n$ such that i induces isomorphisms

$$\pi_q Y_n \cong \begin{cases} \pi_q X & 0 \leq q \leq n \\ 0 & q \geq n+1. \end{cases}$$

Proposition 2.3. *Given any X, there exists a CW complex ΓX and a weak equivalence $\Gamma X \xrightarrow{\gamma} X$.*

Proof. This proof is a variation of the proof in *Concise* that Peter May suggested to me.

Assume X is path connected or do it over each path component. Choose a set or representatives $J = \{j_q \mid [j] \in \pi_q X,\ q \geq 1\}$.

Let $X_1 = \bigvee_{j_q \in J} S^q$ with the standard CW structure and $\gamma_1 = \bigvee j_q : X_1 \to X$. By construction, γ_1 is a 1–equivalence (both spaces are connected and γ_1 induces a surjection on π_*.)

Suppose that X_n has been constructed so that $\gamma_n : X_n \to X$ is an n–equivalence. Choose a set or representatives

$$J = \{j \mid [j] \in \pi_n X_n,\ [\gamma_n \circ j] = 0 \in \pi_n X\}.$$

For each $j \in J_n$, let h_j be an extension of $S^n \xrightarrow{\gamma_n \circ j} X$ to D^{n+1}. Then as in the warm-up above, construct X_{n+1} as the pushout and γ_{n+1} as the canonical map:

$$\begin{tikzcd} \sqcup_J S^n \arrow[r, "\sqcup j"] \arrow[d] & X_n \arrow[d] \arrow[ddr, bend left, "\gamma_n"] & \\ \sqcup_J D^{n+1} \arrow[r] \arrow[drr, bend right, "\sqcup h_j"'] & X_{n+1} \arrow[dr, dotted, "\gamma_{n+1}"] & \\ & & X \end{tikzcd}$$

Any map $S^q \to X_{n+1}$ for $q \leq n$ factors through X_n, so $\pi_q\gamma_{n+1}$ factors through $\pi_q\gamma_n$ for $q \leq n$. Since, $X^n \to X^{n+1}$ is an n-equivalence, this implies that $\pi_q\gamma_{n+1} = \pi_q\gamma_n$ is an isomorphism for $0 \leq q < n$. For $q = n$, we have that

$$\begin{tikzcd} \pi_n X^n \arrow[r, two heads] \arrow[dr, "\pi_n\gamma_n"'] & \pi_n X^{n+1} \arrow[d, "\pi_n\gamma_{n+1}"] \\ & \pi_n X \end{tikzcd}$$

and further, any map $j \in \pi_n X^n$ such that $[\gamma_n \circ j] = 0 \in \pi_n X$ extends to D^{n+1} in X_{n+1}, hence maps to zero in $\pi_n X_{n+1}$. Therefore, $\pi_n\gamma_{n+1}$ is also injective, thus an isomorphism.

Then $\Gamma X = \operatorname{colim}_n X_n$ and $\gamma = \operatorname{colim}_n \gamma_n$ have the desired property. □

Remark 2.4. If X is $n-1$–connected, by construction, ΓX has no q–cells for $1 \leq q \leq n-1$.

We call a map $X' \to X$ which is a weak equivalence with X' a CW-comlex a *CW-approximation* of X.

Theorem 2.5. *Given any map $X \xrightarrow{f} Y$ and CW-approximations of X and Y, there is a cellular map f' and a diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

which commutes up to homotopy. Further, f' is unique up to homotopy.

Proof. Since X' is a CW complex and $Y' \to Y$ is a weak equivalence, by Whitehead's theorem,

$$[X', Y'] \to [X', Y]$$

is a bijection. Therefore, $X' \to X \xrightarrow{f} Y$ has a unique lift up to homotopy, and we can choose a cellular representative f'. □

Remark 2.6. We can also do this for pairs. Let A be a subspace of X.

(1) If $\Gamma A \to A$ is a CW–approximation, then there is a CW-approximation $\Gamma X \to X$ with ΓA a subcomplex of ΓX.

(2) Given a map of pair $(X, A) \to (Y, B)$ and any CW–approximations (X', A') and (Y', B') by CW–pairs, there is a cellular map of pairs f' and a diagram

$$\begin{tikzcd} (X', A') \arrow[r, "f'"] \arrow[d] & (Y', B') \arrow[d] \\ (X, A) \arrow[r, "g"] & (Y, B) \end{tikzcd}$$

which commutes up to homotopy. Further, f' is unique up to homotopy.