MATH 6280 - CLASS 2

Contents

1. Categories

Definition 1.1. A *category* is

- a collection of *objects* $\mathrm{obj}(\mathcal{C})$
- for any two objects $X, Y \in \mathcal{C}$, a set of *morphisms* $\mathcal{C}(X, Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$
- For every object $X \in \mathcal{C}$, an *identity* morphism $\mathrm{id}_X = 1_X \in \mathcal{C}(X, X)$
- For any $X, Y, Z \in \mathcal{C}$ composition law:

$$\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

that satisfy the following properties:

- $\circ$ is associative: $h \circ (g \circ f) = (h \circ g) \circ f$
- $\circ$ is unital: $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f$ for $f \in \mathcal{C}(X, Y)$.

A morphism $f \in \mathcal{C}(X, Y)$, is *invertible* if there exists $g \in \mathcal{C}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Example 1.2.
(1) The category of sets Sets with set functions.
(2) The category of groups Gr with group homomorphisms.
(3) The category of abelian groups Ab with group homomorphisms.
(4) The category of topological spaces Top with continuous functions.
(5) The homotopy category of topology spaces hTop with morphisms homotopy classes of maps $[X, Y]$.
(6) The following is a category I with two objects an one non-identity morphism:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[loop above] & \bullet \arrow[loop above] \end{tikzcd}$$

(7) From any directed graph, you can form a category by adding the identity morphisms and compositions.

(8) The category associated to a poset P with object elements $p \in P$ and one morphisms $p \to q$ if $p \leq q$.

(9) The category $\mathrm{Vect}_{\mathbb{F}}$ of vector spaces with morphisms linear transformations.

(10) There is a canonical way to make a monoid M into a category B_M with one object $\bullet$ where $\mathrm{Hom}_{B_M}(\bullet, \bullet) = M$.

(11) To any category $\mathcal{C}$, there is an opposite category $\mathcal{C}^{op}$ where $\mathrm{obj}(\mathcal{C}^{op}) = \mathrm{obj}(\mathcal{C})$ but $\mathcal{C}^{op}(X, Y) = \mathcal{C}(Y, X)$ (so, you flip all the arrows).

Definition 1.3. A category $\mathcal{C}$ is small if it has a set of objects.

Definition 1.4. A small category $\mathcal{C}$ is a groupoid if all of its morphisms are invertible. For example G is a group, then the category B_G is a groupoid.

Example 1.5. The fundamental groupoid $\Pi(X)$ of a space X is the category whose objects are points of X and morphisms $\Pi(X)(x, y)$ from x to y are paths from $x \to y$ modulo homotopy equivalences which fix the end points. Therefore,

$$\Pi(X)(x, x) = \pi_1(X, x).$$

2. Functors

Definition 2.1. A functor is a *morphism* between categories. That is, a *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a map which sends an object $X \in \mathcal{C}$ to an object $F(X) \in \mathcal{D}$ and a morphism $f : X \to Y$ in $\mathcal{C}$ to a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
- $F(g \circ f) = F(g) \circ F(f)$

A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$ (this is a functor that switches the direction of arrows).

Remark 2.2. Given $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, one can form the functor composite $GF = G \circ F$.

Example 2.3. (1) There is an identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

(2) The is a *forgetful* functor $U : \mathrm{Top} \to \mathrm{Sets}$.

(3) There is a *discrete* space functor $F : \mathrm{Sets} \to \mathrm{Top}$.

(4) There is a *forgetful* or *underlying* functor $U : \mathrm{Ab} \to \mathrm{Sets}$ which sends A to the set underlying A.

(5) There is a free abelian group functor $F : \mathrm{Sets} \to \mathrm{Ab}$ which sends a set S to the free abelian group $F(S)$ generated by S.

(6) Homology $H_n : \mathrm{Top} \to \mathrm{Ab}$ which sends a space S to the n'th simplicial homology group of S.

(7) Cohomology $H^n : \mathrm{Top}^{op} \to \mathrm{Ab}$ which sends a space S to the n'th simplicial cohomology group of S.

(8) The homotopy groups functors: $\pi_n : \mathrm{Top}^{op} \to \mathrm{Ab}$

(9) If $F : G \to H$ is a group homomorphism, then it gives rise to a functor on the associated categories: $\mathcal{F} : B_G \to B_H$.

(10) A functor $F : B_G \to \mathrm{Sets}$ is the same as a set $X = F(\bullet)$ with a group action: i.e. for each g, an element $F(g) \in \mathrm{Sets}(X, X)$.

Definition 2.4. Let $\mathcal{C}$ be any category, and $X \in \mathcal{C}$. A functor $F : \mathcal{C} \to \mathrm{Sets}$ is *representable* by X if it is of the form

$$F(Y) \cong \mathcal{C}(X, Y)$$

for every $Y \in \mathcal{C}$. A functor $G : \mathcal{C}^{op} \to \mathrm{Sets}$ is *representable* if it is of the form

$$G(Y) \cong \mathcal{C}(Y, X).$$

(These isomorphisms have to be *natural* in a sense that we will see below.)

Example 2.5. The functor $U : \mathrm{Top} \to \mathrm{Sets}$ is corepresentable by $*$.

$$U(X) = \mathrm{Top}(*, X).$$

3. Natural Transformation

Definition 3.1. A *natural transformation* is a morphism of functors. That is, if $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\eta : F \to G$ is a collection of morphisms $\eta_X : F(X) \to G(X)$ which make the following diagrams commute for every $f : X \to Y$ in $\mathcal{C}$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y). \end{tikzcd}$$

If each η_X is an isomorphism, then η is a *natural isomorphism* and we write $F \cong G$.